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BERKELEY, CALIFORNIA

WATER WAVES. I.

by

John V. Wehausen

Under Contract Number N-onr-222(30)

SERIES NO. 82
ISSUE NO. 6
DATE August, 1958 (revised July 1959)

ERRATA AND ADDENDA

Vol. I

p. 2a, l. 3*: Read "Villat" for "Villet."

p. 23, l. 2: The end of the line should read, "where $y=o$ is."

p. 88, formula (13:40): In the first term read $z-c$ for $z-o$.

Vol. II

p. 147, l. 13: Replace the sentence beginning "Weinstein ...." by "Completeness follows from known criteria [see, e.g., N. Levinson, Gap and density theorems, Amer. Math. Soc. Colloq. Publ. no. 27, New York, 1940, ch. I]. However, both orthogonality and completeness are consequences of the general theory of Sturm-Liouville systems.

p. 148, formula (16.5): The limits of the integrals with respect to y should be $-\infty$ and 0 instead of 0 and ∞ .

p. 150, l. 7*: Read 23a for 22.

p. 154, l. 12: Read $e^{-i\sigma t}$ for $e^{i\sigma t}$.

p. 156, l. 7: The line should begin "in terms of the wave length and the conformal mapping....."

p. 156, l. 2*: The line should begin "Ursell, and also Marnyanskii [1954], also ....."

p. 157, l. 9: Add to this paragraph the following sentence: "Bartholomeusz [1958] treats the long-wave approximation for reflection at a step in the bottom."

p. 157, l. 13, 14:	These two lines should read, "using the completeness of the functions [cf. (16.1)]."
p. 159, l. 12:	For "value" read "value (see Kreisel [1949] or Meyer [1955])."
p. 160, l. 3:	Insert the following sentence between this formula and line 4: "Other general considerations will be found in Biesel and LaMéhauté [1955]."
p. 161, l. 5-7:	Read "Dean [1945, 1946]" for "Dean [1945]." In lines 5 and 6 delete the clause "the .... $\pi/2n$." In line 7 read "[1957]" for "[1958]."
p. 161, l. 10:	Add the following to this paragraph: "Dean and John also treat barriers inclined at an angle $\pi/2n$. Levine and Rodemich [1958] solve the vertical-barrier problem by several methods, including the cited ones, and then apply one of them to the problem of waves incident upon two parallel vertical barriers."
p. 167, l. 7*:	Read K_1 for K_2.
p. 175, l. 13:	Complete the sentence by "when the body is completely submerged."
p. 179, l. 6:	Between "again" and "by" insert "by Macdonald [1896], Pocklington [1921] and ." In line 7 read "All" for "Both."
p. 180, formula (17.30):	On the right side read $\bar{f}(\zeta)$ for $f'(\zeta)$.
p. 180, l. 2*:	Read "Lehman" for "Lehmann."
p. 181, l. 4:	Read (17.32) for (17.12).
p. 191, l. 7:	For "enough" read "enough and the body is submerged."
p. 204, l. 2*:	Read "Donn" for "Dunn."
p. 205, l. 8*:	Replace the third sentence of this paragraph by the following: "The solutions (18.33) for $\gamma = \pi/2(2n+1)$ have also been given

by Macdonald [1896]. At these critical angles the solution (18.33) does not vanish as $x \to \infty$. Macdonald apparently discarded the other solutions as being of little interest, not 'being sensible at a distance from the edge.' Roseau [1958] has recently carried through a systematic investigation of edge waves, including ones with singular behavior at the edge."

p. 208, l. 1*:	For "discovered" read "states."
p. 216, l. 7:	Insert the following sentence: "Alblas [1958] treats similar three-dimensional problem in which the motion is periodic along the length of the strip."
p. 216, l. 8*:	For "approximately" read "approximately; by improving the approximation, Levine [1958] has clarified certain anomalous results of Kochin for an oscillating horizontal plate."
p. 216, l. 1*:	Add the following sentence to the paragraph: "A general survey of methods of generating waves in the laboratory, including some account of theoretical results, may be found in a recent paper by Biesel and Suquet [1951, 1952]."
p. 218, formula (19.11):	The first integrand should read $\left[\frac{1}{r}\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right]$.
p. 225, l. 7:	In the space following "breadth" insert "b."
p. 231, l. 4:	Delete the sentence beginning on this line.
p. 239, formula (19.63):	Insert "-" before "Re."
p. 241, l. 6:	For "forces" read "coefficients."
p. 242, l. 11:	Replace "by MacCamy [1954]" by "by Barakat [1958] (in an earlier investigation by MacCamy [1954] the multipole terms in the potential for the diffracted wave were omitted)."

p. 243, l. 4: Replace the rest of the paragraph, starting with "MacCamy," by the following: "Barakat shows that φ° can be found as a series in functions of the form (13.21), with $b=0$ and account taken of certain symmetries, and functions of the form (13.20) with $b=0$ and $m=n$.
Let

$$G_{2k}^{2m} = \left[\frac{P_{2k}^{2m}(\cos\theta)}{r^{2k+1}} - \frac{\nu}{2k-2m}\,\frac{P_{2k-1}^{2m}(\cos\theta)}{r^{2k}}\right]\cos 2m\alpha, \quad k=1,2,\cdots,\ m=0,\cdots,k-1;$$

$$G_{2k}^{2m-1} = \left[\frac{P_{2k+1}^{2m-1}(\cos\theta)}{r^{2k+2}} - \frac{\nu}{2k-2m+2}\,\frac{P_{2k}^{2m-1}(\cos\theta)}{r^{2k+1}}\right]\cos(2m-1)\alpha, \quad k=1,2,\cdots;\ m=1,\cdots,k;$$

$$\Phi_n = \left[\frac{P_n^n(\cos\theta)}{r^{n+1}} + (-1)^n\, PV\!\int_0^\infty \frac{k+\nu}{k-\nu}\, k^n e^{kz} J_n(kR)\,dk + 2\pi i(-1)^n \nu^{n+1} e^{\nu z} J_n(\nu R)\right]\cos n\alpha, \quad n=1,2,\cdots.$$

Then φ° may be expressed as follows

$$\varphi^{\circ} = \sum_{k=1}^{\infty}\sum_{m=0}^{k-1} A_{2k}^{2m} a^{2k+2} G_{2k}^{2m} + i\sum_{k=1}^{\infty}\sum_{m=1}^{k} B_{2k}^{2m-1} a^{2k+3} G_{2k}^{2m-1} + \sum_{n=0}^{\infty} C_n \Phi_n,$$

where the complex coefficients A_{2k}^{2m}, B_{2k}^{2m-1}, C_n are to be determined from (19.75). No numerical computations seem to be available."

p. 247, l. 1, 3, 4, 5: Read "c^2/gh" for "c^2h/g." In line 3 read ">1" for ">0."

p. 248, l. 1*: To the sentence terminating on this line add "(see also Lamb [1934])."

p. 249, l. 14: In the parentheses insert "Kochin [1937 ab, 1938 c]," before "Long."

p. 250, l. 7: Insert "$(x_0, -h, \zeta_0)$" after "point."

p. 250, formula (20.18): Read "$\frac{x - x_0}{\varepsilon} + x_0$" and "$\frac{\zeta - \zeta_0}{\varepsilon} + \zeta_0$" for "$\frac{x}{\varepsilon}$" and "$\frac{\zeta}{\varepsilon}$" respectively.

p. 257, l. 8-9: Read "singularities" for "singularization."

p. 257, l. 11: Read "a single-valued complex" for "the complex."

p. 266, l. 5: Read "[1951, 1958]" for "[1951]."

p. 268, l. 6: To the end of the paragraph add the following: "Cummins [1954] finds the additional effect of a train of waves on the surface."

p. 274, l. 4*: Read "Froude" for "Fronde."

p. 275, l. 7*, 8*: The statement should be weakened as follows: "This is confirmed only approximately in photographs; difficulty.... ."

p. 276, l. 6, 12, 1*: Read "$-a < x < a$" for "$-a \leq x \leq a$." In formula (20.73) the lower limit of integration is "$-a$."

p. 288, l. 9: Add the following to this paragraph: "More recently the problem has been considered again by authors unaware of the earlier work. Squire [1957] has analyzed a gliding flat plate by a method similar to that used by Sretenskii and Maruo. Certain integrals involved in this method have been tabulated by Miller [1957]. Cumberbatch [1958] has used a method similar to Sedov's. Both authors add new results to the earlier work."

p. 301, l. 3: Add "$p = p_1 + j p_2$," to this line.

p. 311, l. 8: Add the following to this paragraph: "Also, in the neighborhood of $\alpha = \alpha_0$ the expressions in (21.36) are inaccurate; in this region η may be expressed in terms of Airy functions (see Ursell [1959])." In lines 5 and 8 read "Fig. 1a" for"Fig. 1."

p. 315, l. 5*: Replace this line by "analogues of (13.49) and (13.53) for pressure distributions may be found in equations (22.48) and (22.49) or in." In line 4* read "these" for "this."

Vol. III

p. 329, l. 7: Read "final" for "find."

p. 322, l. 11*: Read $H(\zeta, t)$ for $H(\gamma, d)$.

p. 341, l. 6: Insert "(see also Wurtele [1955])" following "[1953]."

p. 367, l. 3: Add the following to the paragraph: "A problem somewhat related to those of this section is the motion of a freely floating body in a fixed bounded basin (there is now no dissipation of energy as in the problem treated at the end of section 22γ). This problem has been dealt with by Perzhnyanko [1956] and Moiseev [1958]."

p. 378, l. 2*: Read "ryabi" for "ryaby."

p. 422, l. 11: Read "approximation" for "approximations."

p. 422, l. 1*: Read [1919a] for [1918].

p. 428, formula (27.21): Read $\frac{3}{2}m^2 T'$ for $3m^2 T'$.

p. 430, formula (27.29): Read $\frac{1}{16}$ for $\frac{7}{16}$.

p. 433, l. 3*: Insert "and by De [1955] to the fifth order" following "third order."

p. 437, formula (27.43): Read τ_{av} for $\mathcal{E}_{av}$, to conform with the notation elsewhere.

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This report constitutes the first part of an article on Water Waves being prepared for the new edition of the Encyclopaedia of Physics (Handbuch der Physik) published by Springer Verlag.

INDEX

A. Introduction

The various problems of fluid motion treated in this article have in common the property that the fluid is subject to a gravitational force. In addition, in almost all cases they also have in common the presence of surfaces separating two fluids of different densities or, if only one fluid is present, of so-called free surfaces. However, not all fluid flows falling into this category are treated here: tidal motion is treated in vol. 48 in the article by A. Defant. The observed properties of ocean waves and their generation by wind are treated in the article by H. U. Roll, also in vol. 48. Closely related problems concerning flows with free surfaces are treated in the article by D. Gilbarg in this volume.

The subject of water waves engaged many of the mathematicians and mathematical physicists of the last century. Moreover, the last several years have brought a renewed interest in the theory of water waves. In addition to this extensive literature on theoretical aspects of the subject, there have also been many experimental investigations, usually carried out by hydraulic engineers. Hydraulic engineers have also produced an extensive literature, both theoretical and experimental, on open channel flow, flow over weirs and through sluice-gates, etc.; included is a considerable literature on numerical and graphical methods of solving the equations involved. Oceanographers have produced their own literature, usually emphasizing different aspects of the

subject. The theory of ship waves has produced its own literature.

All this material is pertinent to this article. Clearly some selection has to be made. We have followed roughly the following rules: Fundamental results are derived in full. The treatments of various special problems are selected so as to exemplify particular methods, other methods being mentioned only by literature citation. Experimental results are not usually reproduced, but references are given. Numerical methods of solving equations are not treated at all. The more special problems of hydraulic engineering are also not treated. Geophysical aspects which are omitted have already been mentioned.

Several excellent expositions of the theory of waves or of various parts of it already exist. We mention the following*: Lamb [1932, chs. VIII (pp. 250-362) and IX (pp. 363-475)]; Basset [1888, ch. XVII (pp. 144-187)]; Wien [1900, ch. V (pp. 166-224)]; Kochin, Kibel', and Roze [1948, ch. 8 (pp. 394-526)]; Milne-Thomson [1956, ch. XIV (pp. 374-431)]; Airy [1845]; Bouasse [1924]; Auerbach [1931]; Thorade [19[illegible]]; Sretenskii [1936]; Khristianovich [1938]; Keulegan [1950]; Eckart [1951]; and Stoker [1957]. The last cited book by Stoker gives an up-to-date account of much of the fundamental theory. For observation of waves of many kinds, Cornish [1910, 1934] and Miche [1954] should be consulted. Shuleikin [1953, part 3 (pp. 213-292)] contains a general discussion of

*References are collected at the end and identified in the text by author and date.

topics of interest in oceanography. Russell and Macmillan [1952] give a rather nontechnical discussion of ocean waves. A volume published by the Society of Naval Architects of Japan (Zôsen Kyôkai) contains expository papers on various aspects of water-wave theory related to ships (see Maruo [1957], Jinnaka [1957], Nishiyama [1957], Bessho [1957], and Inui [1957]).

For extensive bibliographies one should consult Thorade [1931, pp. 195-211]; Sretenskii [1936, pp. 294-303]; Kampé de Fériet [1932, pp. 225-229]; and Stoker [1957, pp. 545-560]. Sretenskii [1950, 1951] in a survey of the accomplishments of the USSR during the years 1917-1947 has given a rather complete bibliography of Russian papers during those years. Takao Inui [1954] has included a valuable bibliography of Japanese papers in a survey of Japanese contributions to the theory of ship waves. An interesting early history of the subject may be found in a paper by St.-Venant and Flamant [1887]. The treatise by the Weber brothers [1825] is still of interest for its content, and especially for its many references to and summaries of the early papers on water waves. The section on waves of the article on hydrodynamics by Love [1914], as modified by Appell, Beghin and Villet, in the Encyclopédie des sciences mathématiques gives brief indications of the contents of many of the papers published up to about 1912.

B. Mathematical Formulation

1. Coordinate systems and conventions

In the mathematical description of waves one may, as in fluid mechanics in general, describe the motion by describing either the paths of individual fluid particles ("Lagrangian" description) or the velocity (and acceleration) field in the region occupied by fluid at a given moment ("Eulerian" description). Generally, but not always, the Eulerian description will be used.

Rectangular coordinates may be used conveniently for almost all problems. The y-axis will be taken directed oppositely to the force of gravity, the x-axis and z-axis so as to form a right-handed system (i.e., if the y-axis is toward the top of the page and the x-axis is toward the right, the z-axis will point toward the reader). This is a somewhat unconventional choice for the z-axis, but has the obvious advantage that in two-dimensional problems one can delete z-dependent terms from the equations, have conventional (x,y) coordinates, and set $z=x+iy$ without ambiguity when complex-variable methods are convenient.

It seems hardly worth while to try to formulate rules concerning when a moving coordinate system is preferable to a fixed one. However, use of a moving coordinate system is clearly convenient in those cases where it allows one to formulate a problem in a time-independent manner.

The following well-established convention with regard to use of certain letters will be adhered to. The components of the velocity vector $\mathbf{v}$ will be denoted by u, v, w the pressure by p and the density by ρ. The coefficient of viscosity of the

fluid will be denoted by μ, the coefficient of kinematic viscosity, μ/ρ, by ν. The acceleration resulting from gravity is denoted by g.

In the Eulerian formulation one seeks $\underset{\sim}{v}$, p and ρ as functions of x, y, z, t i.e., at any instant t one seeks a vector function and two scalar functions defined on the region occupied by fluid at that instant. In the Lagrangian system one focuses attention on the trajectories of individual particles in the fluid: if a, b, c are the coordinates of a particle at time $t = 0$, then one seeks the position $x(a,b,c,t)$, $y(a,b,c,t)$, $z(a,b,c,t)$ of this point at a later time t. One may pass from one system to the other by means of the equations

$$\frac{dx}{dt} = u(x,y,z,t), \quad \frac{dy}{dt} = v(x,y,z,t), \quad \frac{dz}{dt} = w(x,y,z,t) \tag{1.1}$$

with $x = a, y = b, z = c$ at $t = 0$ as initial conditions.

2. Equations of motion

Derivations of the fundamental equations describing fluid motion are available in many places (e.g., vol. 8 of this Encyclopedia). The equations are reproduced here for convenience of reference.

The equation of continuity in Eulerian coordinates is

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \qquad (2.1)$$

If the fluid is incompressible, but not necessarily homogeneous, $d\rho/dt = 0$ (but not necessarily $\partial\rho/\partial t = 0$) and eq. (2.1) becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad (2.2)$$

In Lagrangian coordinates this may be written

$$\rho(x, y, z, t)\, D = \rho(a, b, c, 0) \qquad (2.3)$$

where

$$D = \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{vmatrix}$$

For an incompressible fluid $\rho(x, y, z, t) = \rho(a, b, c, 0)$ and (2,3) becomes

$$D = 1 \qquad (2.4)$$

The dynamical equations take different forms according as one does or does not try to take account of viscosity. The Navier-Stokes equations for the motion of an incompressible viscous fluid, when the only external force is that of gravity, are as follows in Eulerian coordinates:

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\Delta u,\\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= -g - \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\mu}{\rho}\Delta v, \qquad (2.5)\\
\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\Delta w.
\end{aligned}$$

If viscosity is neglected, the last two terms on the right side of the equations are to be deleted and one obtains the equations for an "ideal" fluid:

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x},\\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= -g - \frac{1}{\rho}\frac{\partial p}{\partial y}, \qquad (2.6)\\
\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z}.
\end{aligned}$$

In Lagrangian coordinates the latter equations become:

$$\begin{aligned}
\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial a} + \left(g + \frac{\partial^2 y}{\partial t^2}\right)\frac{\partial y}{\partial a} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial a} &= -\frac{1}{\rho}\frac{\partial p}{\partial a},\\
\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial b} + \left(g + \frac{\partial^2 y}{\partial t^2}\right)\frac{\partial y}{\partial b} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial b} &= -\frac{1}{\rho}\frac{\partial p}{\partial b}, \qquad (2.7)\\
\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial c} + \left(g + \frac{\partial^2 y}{\partial t^2}\right)\frac{\partial y}{\partial c} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial c} &= -\frac{1}{\rho}\frac{\partial p}{\partial c}.
\end{aligned}$$

The equations of two-dimensional motion result if one deletes all terms containing z, w, and c.

The motion is called irrotational if it satisfies the additional equations

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0, \quad \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0, \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0, \qquad (2.8)$$

or, in two-dimensional motion,

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0. \qquad (2.8')$$

In the case of irrotational motion there exists a potential function $\Phi(x, y, z, t)$ such that

$$u = \frac{\partial \Phi}{\partial x}, \quad v = \frac{\partial \Phi}{\partial y}, \quad w = \frac{\partial \Phi}{\partial z}. \qquad (2.9)$$

It is a classical theorem of hydrodynamics (cf. Lamb [1932, §§17, 33]) that, if the motion of an inviscid fluid with $\rho = \rho(p)$ is irrotational at any instant, it is so thereafter. In particular, a motion started from rest is irrotational.

If $\rho = \rho(p)$ is the equation of state, the following integral of the equations of motion exists for irrotational motion:

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}(u^2+v^2+w^2) + gy + P = A(t) \tag{2.10}$$

where $P = \int_{p_0}^{p} \rho^{-1} dp$ and $A(t)$ is an arbitrary function of t. If the fluid is incompressible, the usual case in this article, ρ is independent of p and the integral becomes:

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}(u^2+v^2+w^2) + gy + \frac{p-p_0}{\rho} = A(t). \tag{2.10'}$$

In this case one obtains also from (2.2) and (2.9)

$$\Delta \Phi \equiv \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0. \tag{2.11}$$

Even if the motion is not irrotational, there still exists an integral like (2.10) if the motion is steady, the so-called Bernoulli integral:

$$\frac{1}{2}(u^2+v^2+w^2) + gy + P = C \tag{2.10''}$$

Here C is constant along a single streamline:

$$\frac{dx}{dt} = u(x,y,z), \quad \frac{dy}{dt} = v(x,y,z), \quad \frac{dz}{dt} = w(x,y,z),$$

but may vary from one streamline to another.

There will be occasion in the following to treat problems in moving coordinate systems. Let $Oxyz$ be a fixed coordinate system and $\bar{O}\bar{x}\bar{y}\bar{z}$ be a system moving with respect to $Oxyz$ but without rotation. Let $\underline{v}_0$ be the vector $\frac{d}{dt}O\bar{O}$, the velocity of a particle referred to $Oxyz$ be $\underline{v}$ and to $\bar{O}\bar{x}\bar{y}\bar{z}$ be $\bar{\underline{v}}$. Then $\underline{v} = \bar{\underline{v}} + \underline{v}_0$. We shall generally want either to describe the absolute motion $\underline{v}$ with respect to the moving coordinate system $\bar{O}\bar{x}\bar{y}\bar{z}$ or the relative motion $\bar{\underline{v}}$ with respect to this coordinate system. In either case the continuity equation remains the

same in form

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial \bar{x}} + \frac{\partial(\rho v)}{\partial \bar{y}} + \frac{\partial(\rho w)}{\partial \bar{z}} = 0, \quad \rho = \rho(\bar{x}, \bar{y}, \bar{z}, t) \tag{2.12}$$

or

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho \bar{u})}{\partial \bar{x}} + \frac{\partial(\rho \bar{v})}{\partial \bar{y}} + \frac{\partial(\rho \bar{w})}{\partial \bar{z}} = 0, \quad \rho = \rho(\bar{x}, \bar{y}, \bar{z}, t). \tag{2.13}$$

The dynamical equations for an ideal fluid for the absolute motion described in the moving coordinate system are:

$$\begin{aligned}
\frac{\partial u}{\partial t} + (u-u_0)\frac{\partial u}{\partial \bar{x}} + (v-v_0)\frac{\partial u}{\partial \bar{y}} + (w-w_0)\frac{\partial u}{\partial \bar{z}} &= -\frac{1}{\rho}\frac{\partial p}{\partial \bar{x}}, \\
\frac{\partial v}{\partial t} + (u-u_0)\frac{\partial v}{\partial \bar{x}} + (v-v_0)\frac{\partial v}{\partial \bar{y}} + (w-w_0)\frac{\partial v}{\partial \bar{z}} &= -g - \frac{1}{\rho}\frac{\partial p}{\partial \bar{y}}, \\
\frac{\partial w}{\partial t} + (u-u_0)\frac{\partial w}{\partial \bar{x}} + (v-v_0)\frac{\partial w}{\partial \bar{y}} + (w-w_0)\frac{\partial w}{\partial \bar{z}} &= -\frac{1}{\rho}\frac{\partial p}{\partial \bar{z}}.
\end{aligned} \tag{2.14}$$

The dynamical equations for the relative motion are:

$$\begin{aligned}
\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} + \bar{w}\frac{\partial \bar{u}}{\partial \bar{z}} &= -\frac{1}{\rho}\frac{\partial p}{\partial \bar{x}} - \dot{u}_0, \\
\frac{\partial \bar{v}}{\partial t} + \bar{u}\frac{\partial \bar{v}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}} + \bar{w}\frac{\partial \bar{v}}{\partial \bar{z}} &= -g - \frac{1}{\rho}\frac{\partial p}{\partial \bar{y}} - \dot{v}_0, \\
\frac{\partial \bar{w}}{\partial t} + \bar{u}\frac{\partial \bar{w}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{w}}{\partial \bar{y}} + \bar{w}\frac{\partial \bar{w}}{\partial \bar{z}} &= -\frac{1}{\rho}\frac{\partial p}{\partial \bar{z}} - \dot{w}_0.
\end{aligned} \tag{2.15}$$

Let us suppose that the motion is irrotational and let $\Phi(x, y, z, t)$ be the velocity potential for the absolute motion in the fixed coordinate system. Let

$$\Phi(x,y,z,t) = \phi\left(\bar{x} + \int^t u_0\,dt,\ \bar{y} + \int^t v_0\,dt,\ \bar{z} + \int^t w_0\,dt,\ t\right) = \bar{\Phi}(\bar{x}, \bar{y}, \bar{z}, t).$$

Then $\bar{\Phi}$ is the velocity potential for the absolute motion in the moving coordinate system:

$$\frac{\partial \bar{\Phi}}{\partial \bar{x}} = u, \quad \frac{\partial \bar{\Phi}}{\partial \bar{y}} = v, \quad \frac{\partial \bar{\Phi}}{\partial \bar{z}} = w.$$

The integral (2.10) becomes:

$$\frac{\partial \bar{\Phi}}{\partial t} + \frac{1}{2}\left[(u-u_0)^2 + (v-v_0)^2 + (w-w_0)^2\right] + g\bar{y} + P = \bar{A}(t), \tag{2.16}$$

where $\bar{A}(t) = A(t) + \frac{1}{2}(u_0^2 + v_0^2 + w_0^2) - g\int^t v_0\,dt$. If one defines $\bar{\bar{\Phi}}$ by

$$\bar{\bar{\Phi}}(\bar{x}, \bar{y}, \bar{z}, t) = \bar{\Phi}(\bar{x}, \bar{y}, \bar{z}, t) - u_0\bar{x} - v_0\bar{y} - w_0\bar{z},$$

then $\bar{\Phi}$ is the velocity potential for the relative motion:

$$\frac{\partial \bar{\Phi}}{\partial \bar{x}} = \bar{u}, \quad \frac{\partial \bar{\Phi}}{\partial \bar{y}} = \bar{v}, \quad \frac{\partial \bar{\Phi}}{\partial \bar{z}} = \bar{w},$$

and the integral (2.10) may be written:

$$\frac{\partial \bar{\Phi}}{\partial t} + \dot{u}_0 \bar{x} + \dot{v}_0 \bar{y} + \dot{w}_0 \bar{z} + \frac{1}{2}(\bar{u}^2 + \bar{v}^2 + \bar{w}^2) + g\bar{z} + P = \bar{A}(t). \tag{2.17}$$

The more general equations when the system $\bar{O}\bar{x}\bar{y}\bar{z}$ is also rotating will not be necessary for this article.

3. Boundary conditions at an interface

Let us now suppose that we are given two immiscible fluids with a common boundary surface, $S(t)$. The one fluid, with density ρ_1 and viscosity μ_1, will occupy region $R_1(t)$; the other, with density ρ_2 and viscosity μ_2, the region $R_2(t)$. Let $F(x,y,z,t)=0$ describe the surface $S(t)$; we assume $F_x^2 + F_y^2 + F_z^2 > 0$ (where $F_x = \partial F/\partial x$, etc.).

The first condition which the surface $S(t)$ must satisfy is a kinematic one. As the surface moves, the velocity of a point (x,y,z) on the surface in the direction of the normal to the surface is given by $-F_t/\sqrt{F_x^2+F_y^2+F_z^2}$. Here one takes the normal in the direction (F_x, F_y, F_z). A particle of fluid at the same point of the surface at that instant will have a velocity component in the direction of the surface normal given by $\dfrac{uF_x + vF_y + wF_z}{\sqrt{F_x^2+F_y^2+F_z^2}} = V_n$. For $S(t)$ to be a bounding surface means, of course, that there can be no transfer of matter across the surface. Consequently the following equation must be satisfied:

$$uF_x + vF_y + wF_z = -F_t, \tag{3.1}$$

where we have used the assumption $F_x^2+F_y^2+F_z^2 > 0$ in dropping the denominators. If one defines the "material derivative"

by the equation

$$\frac{DF}{Dt} = uF_x + vF_y + wF_z + F_t ,$$

then (3.1) is the same as

$$\frac{DF}{Dt} = 0. \tag{3.1'}$$

This condition must be satisfied by any bounding surface, whether an interface or a rigid boundary*.

There are further dynamical conditions to be satisfied at an interface. Let us first consider the general case of viscous fluids with surface tension at the interface. The following assumptions are made:

1) The effect of surface tension as one passes through the interface is to produce a discontinuity in the normal stress proportional to the mean curvature of the boundary surface.

2) For viscous fluids the tangential stress must be continuous as one passes through the interface.

3) For viscous fluids the tangential component of the velocity must be continuous as one passes through the interface.

In order to formulate these statements in mathematical language, we introduce the following notation. Let $g(x,y,z)$ be some function defined in both R_1 and R_2 and let (x_0,y_0,z_0) be a point of the interface S. Assuming that the following limit exists, we shall write $g_r(x_0,y_0,z_0) = \lim g(x,y,z)$ as

*For further discussion of this condition see Truesdell, Bull. Tech. Univ. Istanbul 3 (1950), no. 1, 71-78 (1951); Lichtenstein, Grundlagen der Hydromechanik, Springer, Berlin, 1929, pp. 159-170, 234 ff.

$(x,y,z) \to (x_0,y_0,z_0)$, (x,y,z) in R_i, and $[g(x_0,y_0,z_0)] = g_2(x_0,y_0,z_0) - g_1(x_0,y_0,z_0)$.

Let the components of the stress tensor be denoted by

$$\begin{matrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{matrix}$$

Consider an element of area of the surface S at a point (x,y,z) of S. Let the unit normal vector to S at (x,y,z) be (l,m,n). Then the surface element will have associated with it the stress vector with components:

$$\sigma_{xx} l + \sigma_{xy} m + \sigma_{xz} n, \quad \sigma_{yx} l + \sigma_{yy} m + \sigma_{yz} n, \quad \sigma_{zx} l + \sigma_{zy} m + \sigma_{zz} n.$$

Let R_1 and R_2 be the principal radii of curvature of S at (x,y,z). Then statements 1) and 2) are combined in the one equation

$$\begin{aligned} [\sigma_{xx} l + \sigma_{xy} m + \sigma_{xz} n] &= T(R_1^{-1} + R_2^{-1})\, l, \\ [\sigma_{yx} l + \sigma_{yy} m + \sigma_{yz} n] &= T(R_1^{-1} + R_2^{-1})\, m, \\ [\sigma_{zx} l + \sigma_{zy} m + \sigma_{zz} n] &= T(R_1^{-1} + R_2^{-1})\, n, \end{aligned} \tag{3.2}$$

where T is a constant of proportionality depending upon the two fluids (and their temperatures, but this will not be considered here). T is called the coefficient of surface tension*.

The kinematic condition imposed in (3.1) is clearly equivalent to continuity of the normal component of the velocity as one passes through S. Consequently, the condition 3) above may be combined with this to give

$$u_1 = u_2, \quad v_1 = v_2, \quad w_1 = w_2. \tag{3.3}$$

*For an air-water interface $T = 72.8$ dynes/cm. at 20° C, for mercury-air interface $T = 485$ dynes/cm. at 20° C, for a mercury-water interface $T = 412$ dynes/cm., for benzene-air $T = 28.9$ dynes/cm. at 20° C, for liquid helium-helium vapor $T = 0.24$ dynes/cm. at -270° C.

In the linearized theory of viscosity the stress tensor is given by

$$\begin{matrix} p-\lambda(u_x+v_y+w_z)-2\mu u_x & -\mu(u_y+v_x) & -\mu(u_z+w_x) \\ -\mu(v_x+u_y) & p-\lambda(u_x+v_y+w_z)-2\mu v_y & -\mu(v_z+w_y) \\ -\mu(w_x+u_z) & \mu(w_y+v_z) & p-\lambda(u_x+v_y+w_z)-2\mu w_z . \end{matrix} \tag{3.4}$$

If the fluid is incompressible, the terms containing the second viscosity coefficient λ vanish.

The geometric quantity $R_1^{-1} + R_2^{-1}$ is given by the formula*

$$\frac{1}{R_1}+\frac{1}{R_2} = -\frac{\partial}{\partial x}\frac{F_x}{\sqrt{F_x^2+F_y^2+F_z^2}} - \frac{\partial}{\partial y}\frac{F_y}{\sqrt{F_x^2+F_y^2+F_z^2}} - \frac{\partial}{\partial z}\frac{F_z}{\sqrt{F_x^2+F_y^2+F_z^2}} \tag{3.5}$$

$$= -\frac{F_{xx}(F_y^2+F_z^2)+F_{yy}(F_x^2+F_z^2)+F_{zz}(F_x^2+F_y^2)-2(F_{xy}F_xF_y+F_{yz}F_yF_z+F_{zx}F_zF_x)}{[F_x^2+F_y^2+F_z^2]^{3/2}}.$$

The sign is so selected that, if it is positive, the direction of increase of the normal component of the stress vector at the interfact is in the direction

$$(l,m,n) = \left(\frac{F_x}{\sqrt{F_x^2+F_y^2+F_z^2}}, \frac{F_y}{\sqrt{F_x^2+F_y^2+F_z^2}}, \frac{F_z}{\sqrt{F_x^2+F_y^2+F_z^2}}\right). \tag{3.6}$$

In the case of a surface given by $y=\eta(x,z)$ equation (3.5) becomes

$$\frac{1}{R_1}+\frac{1}{R_2} = \frac{\eta_{xx}(1+\eta_z^2)+\eta_{zz}(1+\eta_x^2)-2\eta_{xz}\eta_x\eta_z}{(1+\eta_x^2+\eta_z^2)^{3/2}}. \tag{3.5'}$$

In the case of two-dimensional motion this simplifies further to the well-known formula

$$\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}. \tag{3.5''}$$

If one now substitutes (3.4), (3.5), and (3.6) in (3.2), one obtains the general boundary condition at the interface.

*See, e.g., A. Duschek and W. Mayer, Lehrbuch der Differentialgeometrie, Bd. I. Teubner, Leipzig-Berlin, 1930, pp. 150-152.

The result is unwieldy in its general form*.

If the interface is given by $y=\eta(x,z)$, the boundary condition becomes

$$
\begin{aligned}
&[p]\eta_x - \{2[\mu u_x]\eta_x - [\mu(u_y+v_x)] + [\mu(u_z+w_x)]\eta_z\} = T(R_1^{-1}+R_2^{-1})\eta_x\\
&[p] + \{[\mu(v_x+u_y)]\eta_x - 2[\mu v_y] + [\mu(v_z+w_y)]\eta_z\} = T(R_1^{-1}+R_2^{-1}) \qquad (3.7)\\
&[p]\eta_z - \{[\mu(w_x+u_z)]\eta_x - [\mu(w_y+v_z)] + 2[\mu w_z]\eta_z\} = T(R_1^{-1}+R_2^{-1})\eta_z
\end{aligned}
$$

with $R_1^{-1}+R_2^{-1}$ given by (3.5'). Here fluid$_1$ is the lower and fluid$_2$ the upper fluid. For two-dimensional motion the equations take the following form:

$$
\begin{aligned}
&[p]\eta'(x) - \{2[\mu u_x]\eta'(x) - [\mu(u_y+v_x)]\} = T\frac{\eta''(x)}{(1+\eta'(x)^2)^{3/2}}\eta'(x),\\
&[p] + \{[\mu(u_y+v_x)]\eta'(x) - 2[\mu v_y]\} = T\frac{\eta''(x)}{(1+\eta'(x)^2)^{3/2}}. \qquad (3.8)
\end{aligned}
$$

One may also write this condition in terms of the components of the stress vector normal and tangential to the interface:

$$
\begin{aligned}
&[p] - 2\,\frac{[\mu u_x]\eta'^2 - [\mu(u_y+v_x)]\eta' + [\mu v_y]}{(1+\eta'^2)^{1/2}} = -T\frac{\eta''(x)}{(1+\eta'^2)^{3/2}},\\
&\frac{2[\mu(u_x-v_y)]\eta' + [\mu(u_y+v_x)](\eta'^2-1)}{(1+\eta'^2)^{1/2}} = 0. \qquad (3.8')
\end{aligned}
$$

If surface tension is to be neglected, one obtains the resulting boundary condition by setting $T=0$ in the various equations above. In this case equation (3.2) simply states the continuity of the stress vector as one passes through the interface.

If viscosity is neglected, but not necessarily surface tension, the condition on the stress vector becomes simply

$$[p] = T(R_1^{-1}+R_2^{-1}), \qquad (3.9)$$

where, of course, the mean curvature is still given by (3.5).

*In tensor notation the condition is somewhat more perspicuous:

$$\{[p]\delta_{ij} - [\lambda u_{k,k}]\delta_{ij} - [\mu(u_{i,j}+u_{j,i})]\}\frac{F_{,j}}{(F_{,k}F_{,k})^{1/2}} = T\,\frac{F_{,r}F_{,s}F_{,rs} - F_{,r}F_{,r}F_{,ss}}{(F_{,k}F_{,k})^{3/2}}\cdot\frac{F_{,i}}{(F_{,k}F_{,k})^{1/2}},$$

where $(x_1,x_2,x_3)=(x,y,z)$, $(u_1,u_2,u_3)=(u,v,w)$ and $F_{,i}=\partial F/\partial x_i$. We have refrained from using tensor notation because its particular advantages cannot in general be exploited here.

The other boundary condition (3.3) changes more drastically upon neglecting viscosity: Condition 3) stating the continuity of the tangential component of velocity is abandoned. The continuity of the normal component, i.e. (3.1), is still retained, of course. Condition 2) concerning the tangential stress is satisfied vacuously for an inviscid fluid.

So far we have considered the boundary condition at an interface between two fluids. If the second fluid is absent, the boundary surface for the first fluid is called a "free surface". Usually the pressure above a free surface is assumed to be some given function, say $p_2(x,y,z,t)$, of position and time; in most cases it is taken to be a constant, either an assumed atmospheric pressure or zero. The boundary conditions concerning the stress vector at a free surface are slight modifications of those for an interface, and can be obtained by setting $\mu_2 = 0, \lambda_2 = 0$. The result is again somewhat unwieldy in its complete form* For an incompressible fluid it is:

$$
\begin{aligned}
(\bar p - p)F_x + \mu\{2u_x F_x + (u_y+v_x)F_y + (u_z+w_x)F_z\} &= T(R_1^{-1}+R_2^{-1})F_x \\
(\bar p - p)F_y + \mu\{(v_x+u_y)F_x + 2v_y F_y + (v_z+w_y)F_z\} &= T(R_1^{-1}+R_2^{-1})F_y \\
(\bar p - p)F_z + \mu\{(w_x+u_z)F_x + (w_y+v_z)F_y + 2w_z F_z\} &= T(R_1^{-1}+R_2^{-1})F_z .
\end{aligned}
\qquad (3.10)
$$

Here we have written $\bar p$ for p_2 , and μ for μ_1; $\bar p, p, u_x,$ are to be evaluated at $F(x,y,z,t) = 0$

The case with which we shall be chiefly concerned is that of an inviscid fluid without surface tension and with

*In tensor notation it may be written:

$$\{(\bar p - p)\delta_{ij} + \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j}+u_{j,i})\}\frac{F_{,j}}{(F_{,k}F_{,k})^{1/2}} = T\,\frac{F_{,rr}F_{,s}F_{,s} - F_{,r}F_{,s}F_{,rs}}{(F_{,k}F_{,k})^{3/2}}\;\frac{F_{,i}}{(F_{,k}F_{,k})^{1/2}}$$

Here we have written $\bar p$ for p_2 and λ, μ for λ_1, μ_1 . All variable quantities in the braces are, of course, to be evaluated at the free surface $F = 0$.

$p(x,y,z,t) = p_0$, a constant. In this case the boundary condition reduces to the single equation

$$p(x,y,z,t) = p_0 \tag{3.11}$$

on $F(x,y,z,t) = 0$. If the motion is irrotational and incompressible, one may determine p explicitly from (2.10') so that (3.10) becomes

$$\Phi_t + \frac{1}{2}(u^2+v^2+w^2) + gy = 0 \tag{3.11'}$$

to be satisfied on $F(x,y,z,t) = 0$.

In the case of steady motion of an incompressible fluid, the Bernoulli integral (2.10") still exists even if the motion is rotational. Consequently, in certain two-dimensional problems of steady motion in which the free surface is a streamline one continues to have a boundary condition like (3.10"):

$$\frac{1}{2}(u^2+v^2) + gy + p_0/\rho = C \tag{3.11"}$$

to be satisfied on $F(x,y) = 0$.

4. Boundary conditions on rigid surfaces

Let the equation of the rigid surface be given by the equation $G(x,y,z,t) = 0$. Then in the case of an inviscid fluid the condition to be satisfied on $G = 0$ is the same as the kinematic condition (3.1):

$$u\,G_x + v\,G_y + w\,G_z = -G_t \, , \tag{4.1}$$

i.e., the component of velocity of the fluid normal to the surface must equal the velocity of the rigid surface in the direction of its normal.

If the fluid is viscous, it must stick to a solid boundary and move with it without slippage. An equation of the form $G(x,y,z,t) = 0$ is not suitable for formulating this statement in equations (e.g., $x^2+y^2+z^2 = a^2$ does not distinguish

between a rotating and a stationary sphere). Let the surface be given in parametric coordinates by: $x = X(r,s,t)$, $y = Y(r,s,t)$, $z = Z(r,s,t)$, where a given point on the surface corresponds to a given pair of values (r,s). Then the condition for viscous fluids may be written:

$$u = \frac{\partial X}{\partial t}, \quad v = \frac{\partial Y}{\partial t}, \quad w = \frac{\partial Z}{\partial t}. \tag{4.2}$$

If a solid boundary penetrates the free surface (or an interface) of a viscous fluid, there will be some difference in treatment of the boundary condition according as the fluid wets the surface or not. In the case of mercury sloshing in a clean glass basin, the fluid pulls free of the surface as it moves up and down, whereas water in the same basin will continue to adhere to any part of the walls already wetted. Furthermore, if surface tension is taken into account, the angle of contact of the free surface with the solid surface will enter into the boundary condition; in the first case mentioned above the angle may vary according as the liquid is rising or falling along the wall*. Although attempts to prove very general existence theorems for fluid motion would presumably take such complications into account, they are usually neglected in most solutions of special problems, there being indeed little choice in the matter.

5. Other types of boundary surfaces

Geophysical problems sometimes suggest situations in which there is an interface between a fluid and an elastic medium. This may occur, for example, in the study of the

*See, e.g., R. S. Burdon, Surface tension and the spreading of liquids, Cambridge, 1949, pp. 76-82.

effect of ocean waves on the ocean floor, as in Longuet-Higgins' [1950] theory of microseisms. Other possibilities are suggested by wave motion on a body of water covered with an ice sheet or at an interface between two fluids separated by an elastic membrane or plate. In one series of investigations the ice sheet has been assumed broken into pieces small with respect to the prevalent wave lengths. In this case only the density of the ice layer enters into the modified boundary condition (see Peters [1950], Keller and Goldstein [1953], Keller and Weitz [1953], Shapiro and Simpson [1953]). Waves in a thin plate over an infinitely deep fluid have been considered briefly by Landau and Lifshits [1953, pp. 762-3], but with neglect of gravity. Greenhill [1887, p.68; 1916] included gravity.

The kinematic boundary condition (3.1) must always hold. The dynamical conditions will depend upon the nature of the assumptions. The matter will not be further considered here.

C. Preliminary Remarks and Developments

6. Classification of problems

Most of the theory of water waves is concerned either with elucidating some general aspects of wave motion or with predicting the behavior of waves in the presence of some special configuration of interest to oceanographers, hydraulic engineers, or ship designers. Unfortunately, even some of the apparently simplest problems have proved too difficult to solve in their most complete formulation. Approximations have been necessary, and in many cases the problems which have been solved are those which could be solved by the approximate methods in use. An examination of the theory also shows that many of the concepts and definitions are almost inextricably bound up with these methods of approximation, following rather than preceding the making of the approximation.

The nature of the approximations used in treating a particular problem provides a natural way of classifying it. First there are the assumptions concerning the properties of the fluid: viscous or inviscid, compressible or incompressible, surface tension or not. Although assuming the fluid to be inviscid, incompressible, and without surface tension simplifies the equations, they are still not easily manageable, even for the simplest kinds of problems. Other approximations of a different nature are required. These are in a sense mathematical approximations. Their physical significance is not in restricting the nature of the fluid but in restricting the charac-

ter of the waves and the boundary configuration. The kind of mathematical approximation used provides another means of classifying problems, and is the principal one which will be used in this article. There are two principal methods of approximation, explained below in section 10, the infinitesimal-wave approximation and the shallow-water approximation. Thus, the development of these two approximate theories and of the exact theory constitutes the bulk of this article.

7. Progressive waves and wave velocity--Standing waves

It will be convenient to call any motion of a fluid in a gravitational field with a free surface or an interface a <u>wave motion</u>.

If the velocity components, pressure, and free surface or interface may be expressed in the form

$$\underline{v} = \underline{v}(x-ct, y, z), \quad p = p(x-ct, y, z), \quad y = \eta(x-ct, z),$$

respectively, then the wave motion will be said to be a <u>progressive wave</u> traveling in the direction Ox. In this case a change to a moving coordinate system with $x' = x - ct$, $y' = y$, $z' = z$ reduces the motion to steady motion with respect to the moving coordinate system. With respect to the fixed coordinate system the profile of the free surface or interface is being transported without change of form in the direction Ox with velocity c. It might seem reasonable therefore to call c the velocity of propagation of the progressive wave.

However, Stokes [1849; or 1880, pp. 202 ff.] has pointed out that the velocity of propagation of the profile of the

free surface does not by itself give a useful definition of wave velocity. Let the fluid be inviscid, either infinitely deep or with a horizontal bottom, and unlimited otherwise. Now let the whole fluid in the progressive wave described above be transported with velocity C (positive or negative) in the direction Ox. Then the motion will still be consistent with the laws of fluid mechanics, the various parts of the fluid will move the same relatively to each other, but the velocity of propagation of the profile will be arbitrary, depending upon the choice of C. What is required for a useful definition of wave velocity is the velocity of propagation of the profile with respect to a coordinate system fixed in some sense in the fluid.

In the case of an infinitely deep fluid, if the axes may be chosen so that as $y \to -\infty$ the velocity relative to these axes vanishes, then one may reasonably measure the profile velocity with respect to these. If the motion far ahead or far behind the disturbance approaches a uniform velocity (possibly zero), then axes moving with the fluid with this velocity may be used. When the disturbance does not behave thus (as in the case of periodic waves) and when the depth is finite, there is no longer an obvious way to select a set of reference axes.

In order to put the problem somewhat differently, let us assume that the wave motion is given as a steady motion with velocity field $\underline{v}(x,y)$ and free surface $y = \eta(x)$. We wish to find a moving coordinate system $x' = x - u_0 t$, $y' = y$,

so that in some sense the relative motion vanishes on the average. We now have the free surface given by $y' = \eta(x'+u_0 t)$ and the relative velocity by $\underline{v}'(x'+u_0 t, y') = \underline{v}(x'+u_0 t, y') - u_0 \underline{i}$. How is u_0 to be chosen? Stokes made two suggestions. One is to define it by the equation

$$\lim_{a\to-\infty, b\to\infty} \frac{1}{b-a}\int_a^b dx' \int_{-h}^{\eta(x'+u_0 t)} u'(x'+u_0 t, y')\,dy'$$
$$= \lim_{a\to-\infty, b\to\infty} \frac{1}{b-a}\int_a^b dx \int_{-h}^{\eta(x)} [u(x,y)-u_0]\,dy = 0, \tag{7.1}$$

where $y=-h$ is the equation for the bottom. In case the motion is periodic, with period λ, the defining equation may be written

$$\int_0^\lambda dx \int_{-h}^{\eta(x)} [u(x,y)-u_0]\,dy = 0. \tag{7.2}$$

If one notes that the mean depth is given by

$$\lim \frac{1}{b-a}\int_a^b [\eta(x)+h]\,dx \quad \text{or} \quad \frac{1}{\lambda}\int_0^\lambda [\eta(x)+h]\,dx,$$

then one sees that, with h' as mean depth,

$$u_0 h' = Q \tag{7.3}$$

where Q is the average discharge rate per unit width. u_0 is thus defined so that the average discharge rate with respect to the (x', y') coordinate system is zero. u_0 is usually denoted by c'.

Stokes' other suggestion was to define u_0 by

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T u'(x'+u_0 t, y')\,dt = 0 \tag{7.4}$$

or

$$u_0 = \lim_{T\to\infty} \frac{1}{T}\int_0^T u(x'+u_0 t, y')\,dt$$

$$= \lim_{T\to\infty} \frac{1}{u_0 T}\int_{x'}^{x'+u_0 T} u(x,y)\,dx = \lim_{a\to\infty} \frac{1}{a}\int_{x'}^{x'+a} u(x,y)\,dx .$$

If u is periodic in x with period λ, one may write

$$u_0 = \frac{1}{\lambda}\int_{x'}^{x'+\lambda} u(x,y)\,dx. \tag{7.5}$$

In either case, for the definition to be useful u_0 must be independent of x' and y. If u is bounded, it follows easily that $\partial u_0/\partial x' = 0$ for both cases. If the motion is irrotational, $u_y = v_x$ and it follows again that $\partial u_0/\partial y = 0$ if v is bounded. Wave velocity defined in this manner is usually denoted by c. For the two special cases considered earlier, the two definitions coincide.

The definition of wave velocity in cases where the motion cannot be reduced to a steady motion is no longer straightforward. In many cases of interest, the asymptotic behavior of the motion for large positive or negative x allows one to define a wave velocity in a manner similar to that above. In more complicated wave motions one may simply follow the motion of some special phase of the profile, say a crest. This provides, for example, a definition of phase velocity for a cylindrical wave.

A general definition of <u>standing wave</u> is somewhat more awkward to formulate than that for a progressive wave. For the case of a plane wave, the free surface $y = \eta(x,t)$ must be periodic in each of x and t, with

wave length λ and period τ, say. In addition, the curves in the (x,t)-plane represented by $\eta(x,t)=0$, where is the undisturbed surface, must consist of two sets of curves oscillating about the lines $x=\frac{1}{2}n\lambda$ and $t=\frac{1}{2}n\tau$, $n=0,\pm 1,\cdots$. For progressive waves the curves $\eta(x,t)=0$ consist of a single set of straight lines, all parallel to $x-ct=0$. The prototype for the standing wave is the surface defined by, say, $y=\sin 2\pi x/\lambda \cos 2\pi t/\tau$. However, as shown by both Penney and Price [1952b] and by Sekerzh-Zenkovich [1947], neither set of curves $\eta(x,t)=0$ consists of straight lines, or even fixed curves, for standing waves of finite amplitude.

There remains the problem of establishing that progressive and standing waves exist under suitable boundary conditions. For the exact boundary conditions for a perfect fluid, the existence of progressive waves was first established by Levi-Civita [1925] and Nekrasov [1921,1922]. The existence of standing waves satisfying the exact boundary conditions is apparently an open question.

8. Energy

Let $T(t)$ be a region occupied by a perfect fluid with a boundary $S(t)$ represented by

$$F(x,y,z,t)=0,$$

the representation being chosen so that (F_x, F_y, F_z) is in the direction of the exterior normal. The surface $S(t)$ moves independently of the motion of the fluid. It is assumed that $T(t)$ contains no singularities of $\mathbf{V}$ and

that surface tension does not act upon the surface $S(t)$ at any time. The energy of the fluid contained in $T(t)$ is given by

$$E = \iiint_{T(t)} \left[\tfrac{1}{2}\rho(u^2+v^2+w^2) + \rho g z\right] d\tau \tag{8.1}$$

For irrotational motion of an inviscid incompressible fluid, one may use (2.10') and express E by

$$E = \iiint_{T(t)} \left[-p - \rho \frac{\partial \Phi}{\partial t}\right] d\tau$$

(Here Φ has been redefined so that $A(t)$ may be set equal to zero.) One may now compute dE/dt by using the general formula:

$$\frac{d}{dt} \iiint_{T(t)} f(x,y,z,t)\, d\tau = \iiint_{T(t)} f_t(x,y,z,t)\, d\tau + \iint_{S(t)} f(x,y,z,t) \frac{-F_t}{\sqrt{F_x^2+F_y^2+F_z^2}}\, d\sigma .$$

One finds [cf. F. John, 1949, p. 19 ff., which we follow closely here]:

$$\frac{dE}{dt} = \iiint_{T(t)} \rho \operatorname{grad}\Phi \cdot \operatorname{grad}\Phi_t \, d\tau + \iint_{S(t)} [\rho\Phi_t + p] \frac{F_t}{\sqrt{F_x^2+F_y^2+F_z^2}}\, d\sigma$$

$$= \iint_{S(t)} \rho \Phi_t \frac{\partial \Phi}{\partial n}\, d\sigma + \iint_{S(t)} [\rho\Phi_t + p] \frac{F_t}{\sqrt{F_x^2+F_y^2+F_z^2}}\, d\sigma$$

by Green's Theorem and the equation of continuity. Finally,

$$\frac{dE}{dt} = \iint_{S(t)} \left\{ \rho\Phi_t \left[\frac{\partial \Phi}{\partial n} + \frac{F_t}{\sqrt{F_x^2+F_y^2+F_z^2}}\right] + p \frac{F_t}{\sqrt{F_x^2+F_y^2+F_z^2}} \right\} d\sigma . \tag{8.2}$$

We recall that $-F_t/\sqrt{F_x^2+F_y^2+F_z^2}$ is the velocity of $S(t)$ in the direction of the exterior normal. Two cases are

of special interest. If $S(t)$ is a "physical" boundary, i.e., one moving with the fluid, then the first summand vanishes and one finds

$$\frac{dE}{dt} = - \iint_{S(t)} p \frac{\partial \Phi}{\partial n} d\sigma \tag{8.3}$$

[cf. Lamb, Hydrodynamics, p.9, eq.(5)]. If $S(t)$ is a fixed "geometrical' boundary, then $F_t = 0$ and one gets

$$\frac{dE}{dt} = \iint_{S(t)} \rho \Phi_t \frac{\partial \Phi}{\partial n} d\sigma \tag{8.4}$$

If one considers any portion of $S(t)$, then the integral of (8.2) taken over this portion and with a minus sign gives the rate of flow of energy through this portion of $S(t)$. In case a part of $S(t)$ is a physical boundary which is fixed, $\partial \phi/\partial n = 0$ and the flow through this part is zero. The same conclusion holds for any portion of $S(t)$ that is a free surface, for then $p = 0$.

If one has a progressive wave moving to the right with $\Phi(x,y,z,t) = \varphi(x-ct, y, z)$ and takes S as a region in the fixed plane $x = x_0$, then the rate of flow of energy through S in the positive direction is given by

$$\iint_S \rho c \varphi_x^2(x_0-ct, y, z)\, dy\, dz \geq 0, \tag{8.5}$$

i.e., energy always flows in the direction of the wave.

In cases where one is dealing with waves generated by moving bodies, it is frequently possible to choose the region T so that no energy is lost from it, the latter being true only as an average if the motion is periodic in time. As an example, consider a body moving steadily with velocity c in the x-direction in an infinite ocean with horizontal

bottom. In addition to the boundary conditions on the body, free surface, and bottom, we assume that the motion vanishes (in the limit) far ahead and to the sides of the body. The surface $S(t)$ may then be chosen as a plane $M: x-ct-a=0$ far ahead, another plane $N: -(x-ct)+\ell=0$ behind the body, planes R and L: $z=\pm a$ on either side, and the bottom H, the wetted surface of the body B, and the part of the free surface F included between the body and the planes. The energy within this region is clearly constant, and one easily obtains, with $\Phi(x,y,z,t)=\varphi(x-ct,y,z)$.

$$0=-\iint_B p\frac{\partial\varphi}{\partial n}\,d\sigma-\iint_{M+N+R+L}\rho c\frac{\partial\varphi}{\partial x}\frac{\partial\varphi}{\partial n}\,d\sigma+c\iint_M\left[\tfrac{1}{2}\rho(\varphi_x^2+\varphi_y^2+\varphi_z^2)+\rho g y\right]d\sigma$$

$$-c\iint_N\left[\tfrac{1}{2}\rho(\varphi_x^2+\varphi_y^2+\varphi_z^2)+\rho g y\right]d\sigma.$$

Since on B one has $\partial\varphi/\partial n=c\cos(n,x)$, one finds for the first integral, remembering that $\underline{n}$ points into the body,

$$-\iint_B p\frac{\partial\varphi}{\partial n}\,d\sigma=-c\iint_B p\cos(n,x)\,d\sigma=Rc,$$

where R is the force on the body. The parts of the second integral over M, R, L vanish as $a\to\infty$ and similarly for the first summand in the third integral. The terms in $\rho g y$ give

$$\int_{-a}^{a}dz\int_{-h}^{\eta(a,z)}\rho g y\,dy-\int_{-a}^{a}dz\int_{-h}^{\eta(\ell,z)}\rho g y\,dy=\int_{-a}^{a}\tfrac{1}{2}\rho g\left[\eta^2(a,z)-\eta^2(\ell,z)\right]dz$$

which, as $a\to\infty$, converges to

$$-\tfrac{1}{2}\rho g\int_{-\infty}^{\infty}\eta^2(\ell,z)\,dz.$$

One obtains finally

$$R = \frac{1}{2}\rho\int_{-\infty}^{\infty}dz\int_{-h}^{\eta(\ell,z)}[-\varphi_x^2(\ell,y,z)+\varphi_y^2(\ell,y,z)+\varphi_z^2(\ell,y,z)]\,dy + \frac{1}{2}g\rho\int_{-\infty}^{\infty}\eta^2(\ell,z)\,dz. \tag{8.6}$$

This exact formula for resistance will be put into a different form later after linearization of the boundary conditions. Although the plane $x-ct=\ell$ may be taken at any distance behind the body without destroying the validity of (8.6), it is usually convenient to take it so far behind that asymptotic expressions for φ can be used.

If in (8.1) a part of the surface $S(t)$, say $S_1(t)$, is an interface with another fluid with surface tension acting, then the energy is given by

$$E = \iiint_{T(t)}\left[\frac{1}{2}\rho(u^2+v^2+w^2)+\rho g y\right]d\tau + T\iint_{S_1(t)}d\sigma. \tag{8.7}$$

Let $S_1(t)$ be bounded by the curve $C(t)$ given parametrically by $x(\lambda,t), y(\lambda,t), z(\lambda,t)$ and let $S(t)=S_1(t)+S_2(t)$. Then the formula analogous to (8.2) is

$$\frac{dE}{dt} = \iint_{S(t)}\rho\varphi_t\left[\varphi_n+\frac{F_t}{\sqrt{F_x^2+F_y^2+F_z^2}}\right]d\sigma + \iint_{S_2(t)}p\frac{F_t}{\sqrt{F_x^2+F_y^2+F_z^2}}d\sigma$$
$$+\iint_{S_1(t)}\left[p+T\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\right]\frac{F_t}{\sqrt{F_x^2+F_y^2+F_z^2}}d\sigma + T\int_{C(t)}\begin{vmatrix}F_x & F_y & F_z\\ x_t & y_t & z_t\\ x_\lambda & y_\lambda & z_\lambda\end{vmatrix}\frac{d\lambda}{\sqrt{F_x^2+F_y^2+F_z^2}}. \tag{8.8}$$

If $S_1(t)$ is a free surface, then the boundary condition

$$p_0 - p = T\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$$

where p_0 is an assumed constant pressure implies that there is no flux of energy through S_1.

If the motion is two-dimensional, with S_1 given by

$y = \eta(x,t)$, $x_1(t) \le x \le x_2(t)$, then (8.7) becomes

$$E(t) = \iint_{T(t)} \left[\tfrac{1}{2}\rho(u^2+v^2) + \rho g y\right] d\sigma + T\int_{x_1(t)}^{x_2(t)} ds \tag{8.9}$$

and (8.8) becomes

$$\begin{aligned}\frac{dE}{dt} &= \int_{S_2} [\rho\Phi_t\,\Phi_n + \frac{F_t}{\sqrt{F_x^2+F_y^2}}]\,ds + \int_{S_1} p\,\frac{F_t}{\sqrt{F_x^2+F_y^2}}\,ds \\ &- \int_{x_1(t)}^{x_2(t)} \left[p + \frac{T+\eta_{xx}}{[1+\eta_x^2]^{3/2}}\right]\eta_t\,dx + T\,\frac{\eta_x\eta_t}{\sqrt{1+\eta_x^2}}\Bigg|_{x_1}^{x_2} + T\sqrt{1+\eta_x^2}\,x'(t)\Bigg|_{x_1}^{x_2}.\end{aligned} \tag{8.10}$$

If S_1 is a free surface, the integral over S_1 may be dropped by suitably redefining p.

9. <u>Momentum</u>

Expressions for rate of change of momentum may be derived which are analogous to those for rate of change of energy. With

$$\underline{M} = \iiint_{T(t)} \rho\,\underline{v}\,d\tau, \tag{9.1}$$

and otherwise the same notation as in section 8, one finds

$$\begin{aligned}\frac{d\underline{M}}{dt} &= \iint_S \rho\left\{\Phi_t\,\underline{n} + \operatorname{grad}\Phi\,\frac{-F_t}{\sqrt{F_x^2+F_y^2+F_z^2}}\right\} d\sigma \\ &= -\iint_S \left\{(p+\rho g y)\,\underline{n} + \rho\left[\underline{v}\cdot\underline{n} + \frac{F_t}{\sqrt{F_x^2+F_y^2+F_z^2}}\right]\underline{v}\right\} d\sigma \\ &= \iint_S \rho\left\{(\Phi_t + \tfrac{1}{2}v^2)\,\underline{n} - \left[\underline{v}\cdot\underline{n} + \frac{F_t}{\sqrt{F_x^2+F_y^2+F_z^2}}\right]\underline{v}\right\} d\sigma.\end{aligned} \tag{9.2}$$

Here the first line of (9.2) is derived by a direct computation of $d\underline{M}/dt$ with $\underline{v} = \operatorname{grad}\Phi$; the second is derived analogously to (8.2); the third follows directly

by use of (2.10'). Comparison of lines one and three gives the known relation (Levi-Civita):

$$\iint_S \tfrac{1}{2} v^2 \underset{\approx}{n}\, d\sigma = \iint_S (\underset{\approx}{v}\cdot\underset{\approx}{n})\, \underset{\approx}{v}\, d\sigma. \tag{9.3}$$

Note that in (9.2) and (9.3) $S(t)$ may move in an arbitrary manner as long as the region $T(t)$ contains no singularities and only fluid. If the boundary is physical, the terms in square brackets vanish in (9.2); if the boundary is fixed, then $F_t = 0$.

Let $S_0(t)$ be a physical boundary, possibly the surface of a solid body, and $S(t)$ a closed surface containing it. Applying (9.2) to the region of fluid bounded jointly by S_0 and S, one finds

$$\begin{aligned} \underset{\approx}{F}_0 &= \iint_{S_0} (p + \rho g y)\underset{\approx}{n}\, d\sigma \\ &= -\iint_{S_0} \rho(\phi_t \underset{\approx}{n} + \underset{\approx}{v}\cdot\underset{\approx}{n}\, \underset{\approx}{v})\, d\sigma + \iint_S \rho(\tfrac{1}{2}v^2 \underset{\approx}{n} - \underset{\approx}{v}\cdot\underset{\approx}{n}\, \underset{\approx}{v})\, d\sigma. \end{aligned} \tag{9.4}$$

Here $\underset{\approx}{F}_0$ is the hydrodynamic force on S_0 and does not include the hydrostatic force.

If singularities are allowed in the region occupied by fluid, they may be enclosed in spheres of small radius and the formula (9.4) applied to the remaining fluid, with S modified to include the spherical surfaces. If the singularities are isolated sources of strengths m_i at the points $\underset{\approx}{a}_i$, then by shrinking the spheres about the singularities in a customary fashion (cf. Milne-Thomson [1956, pp.448-450]), one obtains the following modification of (9.4):

$$\underset{\approx}{F}_0 = -\iint_{S_0} \rho(\phi_t \underset{\approx}{n} + \underset{\approx}{v}\cdot\underset{\approx}{n}\, \underset{\approx}{v})\, d\sigma + \sum 4\pi\rho m_i \underset{\approx}{v}_i + \iint_S \rho(\tfrac{1}{2}v^2 \underset{\approx}{n} - \underset{\approx}{v}\cdot\underset{\approx}{n}\, \underset{\approx}{v})\, d\sigma, \tag{9.5}$$

where $\underset{\approx}{v}_i$ is the velocity at the point $\underset{\approx}{a}_i$ when the source

at that point is removed. Other modifications may be derived for other types of singularities.

If the velocity field is such that $r^{1+\epsilon}v \to 0$ as $r=\sqrt{x^2+y^2+z^2} \to \infty$ for some $\epsilon > 0$, then the last integral in (9.4) or (9.5) will vanish as S is expanded to infinity, provided the latter can be done without destroying the validity of the formula. In the case of a body moving in a fluid with a free surface, one cannot expand in all directions and must include the contribution of the last integral over the free surface. However, the formulas are still useful in computing the force on an obstacle resulting from waves.

10. Expansion of solutions in powers of a parameter

In their exact form even the simplest problems with surface waves are difficult to solve. If one neglects viscosity and assumes irrotational motion, the problem is reduced to finding solutions of Laplace's equation, which is at least linear in the unknown. However, the problem is still difficult because of the nonlinear boundary condition at the free surface or interface. This lack of linearity deprives one, for example, of the mathematical tool of superposition of solutions; expansion in eigenfunctions or use of Green's functions is not possible.

In order to be able to treat special problems, the equations are approximated by ones which are more tractable. The two principal methods of approximation may each be treated as a perturbation procedure. As was mentioned in section 7, this procedure is not concerned with the assumptions about the nature of the fluid, for example, whether or not vis-

cosity is neglected, but rather with the nature of the motion and its generation. An advantage in using the perturbation procedure is that the assumptions about the motion are displayed in such a way that it is clear how to obtain approximations of higher order. The method has been applied to water-wave problems by Sekerzh-Zenkovich [1947, 1951, 1952], K. Friedrichs [1948], Keller [1948], F. John [1949], Longuet-Higgins [1953b], Peters and Stoker [1957], and others. As used here the method is purely formal, the nature of the convergence of the perturbation series, whether it be uniform, pointwise, asymptotic or what not, being left open. However, for each method of approximation it is possible to point to at least one case in which convergence has been proved: for the infinitesimal-wave approximation, Levi-Civita's [1925] proof of the existence of a periodic wave of permanent type; and for the shallow-water approximation, Friedrichs and Hyers' [1954] proof of the existence of a solitary wave.

To a certain extent the two methods of approximation have different aims. The infinitesimal-wave approximation fits into a general scheme for approximating nonlinear equations and boundary conditions by linear ones (see Souriau [1952] for a discussion). To apply it, one must know a particular exact solution to start with. In addition, one must be able to select a dimensionless parameter (or parameters), say ε , which helps to determine the exact physical problem and is such that the solutions to the exact problems associated with each value of ε approach (in some sense)

the known exact solution when $\varepsilon \to 0$. It is then assumed that the various functions entering into the problem may be expanded into power series in ε. The series are substituted into the equations and boundary conditions and grouped according to powers of ε. The coefficients of each power then yield a sequence of equations and boundary conditions, the coefficients of ε giving the first-order theory, those of ε^2 the second-order theory, etc. As an exact initial solution it is usually most convenient to take either a state of rest or of uniform motion. Various choices of ε will be made in the applications later.

The shallow-water approximation differs in that a change of variable involving the expansion parameter is made initially. This introduces ε into the exact equations. When the power series expansions are introduced into the equations, the resulting equations of the sequence are linear in quantities of the same order, but the equations are too degenerate to determine all these quantities without recourse to the equations of next higher order. This leads to nonlinear equations for the desired functions, but ones of a type which have been intensively investigated. In this case the procedure is perhaps artificial in that the perturbation scheme is devised to lead to a special set of equations for a first-order theory, derived originally by quite different reasoning. However, in doing this it makes clear the nature of the approximation and gives a systematic procedure for finding higher-order approximations. It is instructive, in this connection, to read the usual derivation as given, for example, in Lamb [1932, pp.254-256] or

Stoker [1957, pp. 22-25] (who also gives the one given here).

10x. The infinitesimal-wave approximation

We shall derive the equations of motion and the free-surface or interface boundary conditions for this linearized theory without identifying explicitly the parameter ε used in the expansions. Later on, when specific choices are made, the boundary conditions on certain geometric boundaries associated with the choice of ε will be modified to conform with the linearization.

Consider two incompressible viscous fluids in contact along an interface represented by $y = \eta(x, z, t)$. Quantities referring to the upper fluid have subscript 2, those to the lower fluid subscript 1; the coefficient of surface tension is T. Assume the following expansions in the parameter ε:

$$
\begin{aligned}
\underline{v}_i(x, y, z, t, \varepsilon) &= \varepsilon \underline{v}_i^{(1)} + \varepsilon^2 \underline{v}_i^{(2)} + \cdots, \\
p_i(x, y, z, t, \varepsilon) &= p_i^{(0)} + \varepsilon p_i^{(1)} + \varepsilon^2 p_i^{(2)} + \cdots \\
\eta(x, z, t, \varepsilon) &= \varepsilon \eta^{(1)} + \varepsilon^2 \eta^{(2)} + \cdots .
\end{aligned}
\tag{10.1}
$$

Substitute these expansions in equations (2.2), (2.5), (3.1), (3.3), and (3.7), remembering in addition that formal expansions of the following sort, for example, hold:

$$
\begin{aligned}
u_1(x, \eta(x, z, t), z, t) &= u_1(x, 0, z, t) + \eta u_{1y}(x, 0, z, t) + \cdots \\
&= \varepsilon u_1^{(1)}(x, 0, z, t) + \varepsilon^2 [u_1^{(2)}(x, 0, z, t) + \eta^{(1)} u_{1y}^{(1)}(x, 0, z, t)] + \cdots .
\end{aligned}
$$

Collecting first the terms independent of ε, one finds from (2.5) and (3.3)

$$\operatorname{grad}\left(p_i^{(0)}+\rho_i g y\right)=0\,,\quad p_2^{(0)}(x,0,z,t)=p_1^{(0)}(x,0,z,t). \tag{10.2}$$

Collecting the coefficients of the first power of ε , one finds

$$\frac{\partial u_i^{(1)}}{\partial x}+\frac{\partial v_i^{(1)}}{\partial y}+\frac{\partial w_i^{(1)}}{\partial z}=0\,,\quad i=1,2\,,$$

$$\frac{\partial \underline{v}_i^{(1)}}{\partial t}=-\frac{1}{\rho_i}\operatorname{grad}p_i^{(1)}+\nu_i\Delta\underline{v}_i^{(1)}\,,\quad i=1,2\,,$$

$$u_1^{(1)}(x,0,z,t)=u_2^{(1)}(x,0,z,t)\,,$$

$$v_1^{(1)}(x,0,z,t)=v_2^{(1)}(x,0,z,t)=\eta_t\,(x,z,t) \tag{10.3}$$

$$w_1^{(1)}(x,0,z,t)=w_2^{(1)}(x,0,z,t)$$

$$\mu_1\left(u_{1y}^{(1)}(x,0,z,t)+v_{1x}^{(1)}\right)=\mu_2\left(u_{2y}^{(1)}+v_{2x}^{(1)}\right),$$

$$p_2^{(1)}(x,0,z,t)-p_1^{(1)}-(\rho_2-\rho_1)g\eta^{(1)}-2\left(\mu_2 v_{2y}^{(1)}-\mu_1 v_{1y}^{(1)}\right)=T\left(\eta_{xx}^{(1)}+\eta_{zz}^{(1)}\right),$$

$$\mu_1\left(w_{1y}^{(1)}(x,0,z,t)+v_{1z}^{(1)}\right)=\mu_2\left(w_{2y}^{(1)}+v_{2z}^{(1)}\right).$$

If the upper fluid is replaced by a given atmospheric pressure distribution $\bar{p}(x,z,t)$, then the equations for the lower fluid become (after dropping the subscripts)

$$\operatorname{grad}\left(p^{(0)}+\rho g y\right)=0\,,\quad p^{(0)}(x,0,z,t)=\bar{p}^{(0)}(x,z,t)\,,$$

$$\frac{\partial u^{(1)}}{\partial x}+\frac{\partial v^{(1)}}{\partial y}+\frac{\partial w^{(1)}}{\partial z}=0\,,$$

$$\frac{\partial \underline{v}^{(1)}}{\partial t}=-\frac{1}{\rho}\operatorname{grad}p^{(1)}+\nu\Delta\underline{v}^{(1)}\,, \tag{10.4}$$

$$\eta_t^{(1)}(x,z,t) = v^{(1)}(x,0,z,t),$$

$$u_y^{(1)}(x,0,z,t) + v_x^{(1)} = w_y^{(1)} + v_z^{(1)} = 0$$

$$p^{(1)}(x,0,z,t) - \rho g \eta^{(1)} - 2\mu v_y^{(1)} = -T(\eta_{xx}^{(1)} + \eta_{zz}^{(1)}) + \bar{p}(x,z,t).$$

For convenience we have assumed above that the expansion for η starts with $\varepsilon\eta^{(1)}$. If we had assumed instead $\eta = \eta^{(0)} + \varepsilon\eta^{(1)} + \cdots$, we would have found from (3.1) and (3.7) the equations

$$\eta_t^{(0)} = \eta_x^{(0)} = \eta_z^{(0)} = 0$$

and hence, $\eta^{(0)}$ = const. The zero values of η in the boundary conditions would then be replaced by this constant. Taking the constant equal to zero means that we have taken the undisturbed interface as (x, z)-plane.

The equations above give the linearized equations of motion and boundary conditions at the interface or free surface. If one now proceeds, as we shall not do for this case, to collect coefficients of ε^2, one may obtain the differential equations and boundary conditions for the second-order corrections to be added to the solutions of the linearized equations, and so forth for higher-order corrections. In general the resulting equations are too unwieldy to be useful.

A special case of the linearized equations which is of particular interest is irrotational flow of a perfect fluid. There is then a velocity potential ϕ which we assume has the following expansion:

$$\Phi(x,y,z,t,\varepsilon) = \varepsilon\Phi^{(1)} + \varepsilon^2\Phi^{(2)} + \cdots \tag{10.5}$$

Condition (2.11) becomes

$$\Delta\Phi^{(i)} = 0\,, \quad i = 1, 2, \cdots. \tag{10.6}$$

Let there be two superposed fluids with velocity potentials Φ_1 and Φ_2 describing the motion in each; otherwise the same notation as above. Then condition (3.1) at the interface gives the linearized condition

$$\eta^{(1)}_t(x,z,t) = \Phi^{(1)}_{1y}(x,0,z,t) = \Phi^{(1)}_{2y}(x,0,z,t) \tag{10.7}$$

and condition (3.9), together with (2.10'), gives

$$-\rho_2\Phi^{(1)}_{2t}(x,0,z,t) + \rho_1\Phi^{(1)}_{1t}(x,0,z,t) + (\rho_1-\rho_2)g\eta^{(1)}(x,z,t) = T(\eta^{(1)}_{xx} + \eta^{(1)}_{zz}). \tag{10.8}$$

The further special case when both the upper fluid and surface tension are missing will be dealt with so often later on that we repeat the boundary conditions for it. We allow, however, a pressure distribution on the free surface, $\bar{p}(x,z,t) = \varepsilon\bar{p}^{(1)} + \varepsilon^2\bar{p}^{(2)} + \cdots$. The first-order boundary conditions are

$$\begin{aligned} &\eta^{(1)}_t(x,z,t) - \Phi^{(1)}_y(x,0,z,t) = 0, \\ &g\eta^{(1)}(x,z,t) + \Phi_t(x,0,z,t) + \rho^{-1}\bar{p}^{(1)}(x,z,t) = 0. \end{aligned} \tag{10.9}$$

Eliminating $\eta^{(1)}$ between the last two equations, one gets

$$g\Phi^{(1)}_y(x,0,z,t) + \Phi^{(1)}_{tt}(x,0,z,t) + \rho^{-1}\bar{p}^{(1)}_t(x,z,t) = 0. \tag{10.10}$$

The boundary conditions for the second-order corrections are

not too long to write down:

$$\eta_t^{(2)}(x,z,t) - \Phi_y^{(2)}(x,0,z,t) = \eta^{(1)}\Phi_{yy}^{(1)} - \eta_x \Phi_x - \eta_z \Phi_z ,$$
(10.11)
$$g\eta^{(2)}(x,z,t) + \Phi_t^{(2)}(x,0,z,t) + \rho^{-1}\bar{p}^{(2)}(x,z,t) = -\eta^{(1)}\Phi_{ty}^{(1)} - \tfrac{1}{2}(\mathrm{grad}\,\Phi^{(1)})^2 .$$

Eliminating $\eta^{(1)}$ and $\eta^{(2)}$ from (10.11), one finds a counter-part to (10.10):

$$g\Phi_y^{(2)}(x,0,z,t) + \Phi_{tt}^{(2)} + \rho^{-1}\bar{p}_t^{(2)} = -\frac{\partial}{\partial t}(\mathrm{grad}\,\Phi^{(1)})^2 + (\Phi_t^{(1)} + \rho^{-1}\bar{p}^{(1)})(\Phi_{yy}^{(1)} + \tfrac{1}{g}\Phi_{tty}^{(1)}) \tag{10.12}$$
$$- \rho^{-1}(\Phi_x^{(1)}\bar{p}_x^{(1)} + \Phi_z^{(1)}\bar{p}_z^{(1)}) .$$

Under certain circumstances the next-to-last term will vanish. The boundary conditions for higher-order corrections will not be worked out in detail. However, they are of the form

$$g\Phi_y^{(i)}(x,0,z,t) + \Phi_{tt}^{(i)} + \rho^{-1}\bar{p}_t^{(i)} = A_i\{\Phi^{(1)},\cdots,\Phi^{(i-1)},\bar{p}^{(1)},\cdots,\bar{p}^{(i-1)}\}$$
(10.13)
$$g\eta^{(i)}(x,z,t) + \Phi_t^{(i)}(x,0,z,t) + \rho^{-1}\bar{p}^{(i)} = B_i\{\Phi^{(1)},\cdots,\Phi^{(i-1)},\bar{p}^{(1)},\cdots,\bar{p}^{(i-1)}\},$$

where A_i and B_i are functionals of the functions in brackets, in this case complicated polynomials of the functions and their derivatives evaluated at $y=0$.

It is useful to have the form of the linearized boundary conditions when certain additional assumptions are made.

First, let us suppose that the $(\bar{x},\bar{y},\bar{z})$-coordinate system is moving with velocity $c(t)$ in the x-direction with respect

to the fixed (x, y, z)-coordinate system. Then, from the equation following (2.15) with $\bar{y} = y$, $\bar{z} = z$

$$\Phi_t(x,y,z,t) = \bar{\Phi}_t - c\bar{\Phi}_{\bar{x}}, \quad \Phi_{tt} = \bar{\Phi}_{tt} - 2c\bar{\Phi}_{t\bar{x}} + c^2\bar{\Phi}_{\bar{x}\bar{x}} - \dot{c}\bar{\Phi}_{\bar{x}},$$

and the boundary conditions become

$$g\bar{\eta}^{(1)}(\bar{x},\bar{z},t) + \bar{\Phi}_t^{(1)}(\bar{x},0,\bar{z},t) - c\bar{\Phi}_{\bar{x}}^{(1)}(\bar{x},0,\bar{z},t) + \rho^{-1}\bar{p}^{(1)}(\bar{x},\bar{z},t) = 0$$

$$\begin{aligned} \bar{\Phi}_{tt}^{(1)}(\bar{x},0,\bar{z},t) - 2c\bar{\Phi}_{t\bar{x}}^{(1)} + c^2\bar{\Phi}_{\bar{x}\bar{x}}^{(1)} - \dot{c}\bar{\Phi}_{\bar{x}}^{(1)} & \\ + g\bar{\Phi}_{\bar{y}}^{(1)} + \rho^{-1}\bar{p}_t^{(1)} - c\rho^{-1}\bar{p}_{\bar{x}}^{(1)} &= 0 . \end{aligned} \qquad (10.14)$$

If c is constant and the motion is steady in the moving coordinate system,

$$\Phi(x,y,z,t) = \varphi(x-ct, y, z) = \varphi(\bar{x},\bar{y},\bar{z})$$

and the linearized boundary conditions are

$$\begin{aligned} g\bar{\eta}^{(1)}(\bar{x},\bar{z}) - c\varphi_{\bar{x}}^{(1)}(\bar{x},0,\bar{z}) + \rho^{-1}\bar{p}^{(1)}(\bar{x},\bar{z}) &= 0 \\ g\varphi_{\bar{y}}^{(1)}(\bar{x},0,\bar{z}) + c^2\varphi_{\bar{x}\bar{x}}^{(1)}(\bar{x},0,\bar{z}) - c\rho^{-1}\bar{p}_{\bar{x}}^{(1)}(\bar{x},\bar{z}) &= 0 . \end{aligned} \qquad (10.15)$$

If the motion is steady with respect to a moving coordinate system, one may impose a uniform flow in the opposite direction and then treat the problem as a steady one in an absolute coordinate system, but carrying out the perturbation about the uniform flow. We illustrate this for the case of two-dimensional irrotational flow. Let $\varphi(x,y)$ and $\psi(x,y)$ be the velocity potential and stream function,

respectively, and assume expansions of the form

$$
\begin{aligned}
\varphi(x,y) &= -cx + \varepsilon\varphi^{(1)}(x,y) + \varepsilon^2\varphi^{(2)} + \cdots \\
\psi(x,y) &= -cy + \varepsilon\psi^{(1)}(x,y) + \varepsilon^2\psi^{(2)} + \cdots \\
\eta(x) &= \qquad\quad \varepsilon\eta^{(1)}(x) + \varepsilon^2\eta^{(2)} + \cdots
\end{aligned}
\tag{10.16}
$$

The differential equations $\Delta\varphi = 0$, $\Delta\psi = 0$, $\varphi_x = \psi_y$, $\varphi_y = \psi_x$ become

$$\Delta\varphi^{(1)} = 0,\ \Delta\psi^{(1)} = 0,\ \varphi_x^{(1)} = \psi_y^{(1)},\ \varphi_y^{(1)} = -\psi_x^{(1)}. \tag{10.17}$$

The kinematic condition (3.1) is replaced by

$$\psi(x, \eta(x)) = 0$$

Substituting the expansions (10.16) in this equation and in (3.11'), one finds from the coefficients of :

$$
\begin{aligned}
-c\,\eta^{(1)}(x) + \psi^{(1)}(x,0) &= 0 \\
g\,\eta^{(1)}(x) - c\,\varphi_x^{(1)}(x,0) + \rho^{-1}\bar{p}^{(1)}(x) &= 0.
\end{aligned}
\tag{10.18}
$$

Eliminating $\eta^{(1)}$ and using the third of equations (10.17), one gets

$$g\psi^{(1)}(x,0) - c^2\psi_y^{(1)}(x,0) + c\rho^{-1}\bar{p}^{(1)}(x) = 0. \tag{10.19}$$

Collecting the coefficients of ε^2, one obtains after some manipulation

$$
\begin{aligned}
g\psi^{(2)}(x,0) - c^2\psi_y^{(2)} &= \frac{1}{c}\psi^{(1)}\left[c^2\psi_{yy}^{(1)} - g\psi_y^{(1)}\right] - \frac{1}{2}c\left[\psi_x^{(1)2} + \psi_y^{(1)2}\right], \\
c\,\eta^{(2)}(x) &= \psi^{(2)}(x,0) + \frac{1}{c}\psi^{(1)}\psi_y^{(1)};
\end{aligned}
\tag{10.20}
$$

here we have assumed for simplicity that $\bar{p} = 0$.

10.3. The shallow-water approximation

This approximation has been widely used by hydraulic engineers in the study of open-channel flow and, in a further simplification, is used for the theory of tides. In deriving the equations from the exact ones we shall follow the method of Friedrichs [1948] and Keller [1948]. However, a somewhat different approach to this approximation due to Ursell [1953] is also instructive. Although it is possible to carry through the derivation while taking account of surface tension, this will not be done here. It will be assumed to start with that there are two perfect, incompressible fluids with an interface $y = \eta(x, z, t)$; the bottom fluid is bounded below by a rigid surface $y = \ell(x, z)$. Variables pertaining to the lower fluid have subscript 1, those pertaining to the upper fluid subscript 2. The motion will be assumed irrotational.

Before making an expansion in powers of a parameter, it is essential to make a change of variable in which vertical and horizontal distances are stretched by different amounts. Let m be a scale for horizontal measuremen nd n one for vertical measurement. Define $\varepsilon = n^2/m^2$. Introduce new variables, $\bar{x}, \bar{y}, \bar{z}, \bar{t}$, by the equations

$$\bar{x} = x\sqrt{\varepsilon},\ \bar{y} = y,\ \bar{z} = z\sqrt{\varepsilon},\ \bar{t} = t\sqrt{\varepsilon},\ \bar{u} = u,\ \bar{v} = v\sqrt{\varepsilon},\ \bar{w} = w,\ \bar{p} = p. \tag{10.21}$$

Equations (2.2), (2.6), (2.8), (3.1), (3.9), and (4.1) (with $T = 0$) become:

$$\varepsilon\left(\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{w}}{\partial \bar{z}}\right) + \frac{\partial \bar{v}}{\partial \bar{y}} = 0,$$

$$\begin{aligned}
&\varepsilon\left(\frac{\partial\bar u}{\partial\bar t}+\bar u\,\frac{\partial\bar u}{\partial\bar x}+\bar w\,\frac{\partial\bar u}{\partial\bar\zeta}+\frac{1}{\rho}\frac{\partial\bar p}{\partial\bar x}\right)+\bar v\,\frac{\partial\bar u}{\partial\bar y}=0,\\
&\varepsilon\left(\frac{\partial\bar v}{\partial\bar t}+\bar u\,\frac{\partial\bar v}{\partial\bar x}+\bar w\,\frac{\partial\bar v}{\partial\bar\zeta}+g+\frac{1}{\rho}\frac{\partial\bar p}{\partial\bar y}\right)+\bar v\,\frac{\partial\bar v}{\partial\bar y}=0,\\
&\varepsilon\left(\frac{\partial\bar w}{\partial\bar t}+\bar u\,\frac{\partial\bar w}{\partial\bar x}+\bar w\,\frac{\partial\bar w}{\partial\bar\zeta}+\frac{1}{\rho}\frac{\partial\bar p}{\partial\bar\zeta}\right)+\bar v\,\frac{\partial\bar w}{\partial\bar y}=0,\\
&\frac{\partial\bar w}{\partial\bar y}=\frac{\partial\bar v}{\partial\bar\zeta},\quad \frac{\partial\bar u}{\partial\bar\zeta}=\frac{\partial\bar w}{\partial\bar x},\quad \frac{\partial\bar v}{\partial\bar x}=\frac{\partial\bar u}{\partial\bar y},\\
&\varepsilon\left(\bar u\,\frac{\partial\bar\eta}{\partial\bar x}+\bar w\,\frac{\partial\bar\eta}{\partial\bar\zeta}+\frac{\partial\bar\eta}{\partial\bar t}\right)-\bar v=0 \quad \text{for } \bar y=\bar\eta(\bar x,\bar\zeta,\bar t),\\
&\varepsilon\left(\bar u\,\frac{\partial b}{\partial\bar x}+\bar w\,\frac{\partial b}{\partial\bar\zeta}\right)-\bar v=0 \quad \text{for } \bar y=b(\bar x,\bar\zeta,\bar t),\\
&\bar p_2(\bar x,\bar\eta,\bar\zeta,\bar t)=\bar p_1(\bar x,\bar\eta,\bar\zeta,\bar t),
\end{aligned}\tag{10.22}$$

where $\bar u, \bar v, \bar w, \bar p, \rho$ possess suppressed subscripts 1 and 2 for the lower and upper fluids respectively, except in the last equation.

Now assume expansions of the form

$$\begin{aligned}
\bar{\underline{v}}_i &= \underline{v}_i^{(0)}+\varepsilon\,\underline{v}_i^{(1)}+\varepsilon^2\,\underline{v}_i^{(2)}+\cdots, \quad i=1,2,\\
\bar p_i &= p_i^{(0)}+\varepsilon\,p_i^{(1)}+\varepsilon^2 p_i^{(2)}+\cdots, \quad i=1,2,\\
\bar\eta &= \eta^{(0)}+\varepsilon\,\eta^{(1)}+\varepsilon^2\eta^{(2)}+\cdots,
\end{aligned}\tag{10.23}$$

substitute in the equations of (10.22) and collect according to powers of ε. (We shall henceforth suppress the bars on $\bar x, \bar y, \bar\zeta, \bar t, \bar\eta$) The terms independent of ε give the equations

$$\begin{aligned}
&v_y^{(0)}=0\\
&v^{(0)}u_y^{(0)}=0,\quad v^{(0)}v_y^{(0)}=0,\quad v^{(0)}w_y^{(0)}=0,\\
&w_y^{(0)}=v_\zeta^{(0)},\quad u_\zeta^{(0)}=w_x^{(0)},\quad v_x^{(0)}=u_y^{(0)},\\
&v^{(0)}(x,\eta^{(0)},\zeta,t)=0,\quad v_1^{(0)}(x,b,\zeta,t)=0,\\
&p_2^{(0)}(x,\eta^{(0)},\zeta,t)=p_1^{(0)}(x,\eta^{(0)},\zeta,t).
\end{aligned}\tag{10.24}$$

The first and fourth equations give

$$v^{(0)}(x, y, z, t) \equiv 0 \tag{10.25}$$

The third then states that

$$u_y^{(0)} = w_y^{(0)} = 0 \quad \text{or} \quad u^{(0)} = u^{(0)}(x, z, t) \quad w^{(0)} = w^{(0)}(x, z, t). \tag{10.26}$$

The terms which are coefficients of ε give, after making use of (10.25) and (10.26),

$$\begin{aligned}
&u_x^{(0)} + w_z^{(0)} + v_y^{(1)} = 0,\\
&u_t^{(0)} + u^{(0)} u_x^{(0)} + w^{(0)} u_z^{(0)} + p_x^{(0)}/\rho = 0,\\
&g + p_y^{(0)}/\rho = 0,\\
&w_t^{(0)} + u^{(0)} w_x^{(0)} + w^{(0)} w_z^{(0)} + p_z^{(0)}/\rho = 0,\\
&u^{(0)} \eta_x^{(0)} + w^{(0)} \eta_z^{(0)} + \eta_t^{(0)} - v^{(1)} = 0 \quad \text{for } y = \eta^{(0)}(x, z, t),\\
&u_1^{(0)} \ell_x + w_1^{(0)} \ell_z - v_1^{(1)} = 0 \quad \text{for } y = \ell(x, z).
\end{aligned} \tag{10.27}$$

(The equations deriving from irrotationality and the continuity of pressure will be brought in later.) The first and last two equations of (10.27) together with (10.26) give

$$\begin{aligned}
v_1^{(1)} &= -y\left(u_{1x}^{(0)} + w_{1z}^{(0)}\right) + \left(u_1^{(0)} \ell\right)_x + \left(w_1^{(0)} \ell\right)_z,\\
v_2^{(1)} &= -y\left(u_{2x}^{(0)} + w_{2z}^{(0)}\right) + \left(u_2^{(0)} \eta^{(0)}\right)_x + \left(w_2^{(0)} \eta^{(0)}\right)_z - \left(u_1^{(0)} \eta^{(0)}\right)_x - \left(w_1^{(0)} \eta^{(0)}\right)_z\\
&\quad + \left(u_1^{(0)} \ell\right)_x + \left(w_1^{(0)} \ell\right)_z.
\end{aligned} \tag{10.28}$$

The third equation of (10.27) gives

$$p^{(0)} = -\rho g y + f(x, z, t).$$

In order to evaluate f, further information is necessary. Here are two cases of interest. 1) If the upper fluid is absent, the condition $p^{(0)}(x, \eta, z, t) = 0$ gives

$$p^{(0)} = -\rho g y + \rho g \eta^{(0)}(x, z, t) \tag{10.29}$$

2) If the upper fluid is unbounded above, then, up to an additive constant,

$$\begin{aligned} p_1^{(0)} &= -\rho_1 g y + (\rho_1 - \rho_2) g \eta^{(0)} + k, \\ p_2^{(0)} &= -\rho_2 g y + k \end{aligned} \tag{10.30}$$

If the upper fluid is bounded above by a free surface $y = d(x, z, t) = d^{(0)} + \epsilon d^{(1)} + \cdots$, then one may satisfy the boundary condition $p_2^{(0)}(x, d, z, t) = 0$, $p_2^{(0)}(x, \eta, z, t) = p_1^{(0)}(x, \eta, z, t)$ with

$$\begin{aligned} p_1^{(0)} &= -\rho_1 g (y - \eta^{(0)}) + \rho_2 g (d^{(0)} - \eta^{(0)}), \\ p_2^{(0)} &= -\rho_2 g (y - d^{(0)}). \end{aligned} \tag{10.31}$$

It is clear from the form of $p^{(0)}$ why the shallow-water approximation is sometimes called the hydrostatic approximation.

The usual equations for the first approximation to the shallow-water theory are those in which only the lower fluid is present. They may now be obtained by substituting (10.29) in the second and fourth equations in (10.27) and (10.28) in the fifth equation. They are (10.25), (10.29),

and

$$u_t^{(0)} + u^{(0)}u_x^{(0)} + w^{(0)}u_z^{(0)} + g\eta_x^{(0)} = 0,$$

$$w_t^{(0)} + u^{(0)}w_x^{(0)} + w^{(0)}w_z^{(0)} + g\eta_z^{(0)} = 0, \tag{10.32}$$

$$\eta_t^{(0)} + [u^{(0)}(\eta^{(0)}-\ell)]_x + [w^{(0)}(\eta^{(0)}-\ell)]_z = 0.$$

If one now collects the coefficients of ϵ^2 and the remaining coefficients of ϵ , one finds after some reduction

$$u_x^{(1)} + v_y^{(2)} + w_z^{(1)} = 0,$$

$$u_t^{(1)} + u^{(1)}u_x^{(0)} + u^{(0)}u_x^{(1)} + w^{(1)}u_z^{(0)} + w^{(0)}u_z^{(1)} + v^{(1)}u_y^{(1)} + p_x^{(1)}/\rho = 0,$$

$$v_t^{(1)} + u^{(0)}v_x^{(1)} + w^{(0)}v_z^{(1)} + v^{(1)}v_y^{(1)} + p_y^{(1)}/\rho = 0,$$

$$w_t^{(1)} + u^{(1)}w_x^{(0)} + u^{(0)}w_x^{(1)} + w^{(1)}w_z^{(0)} + w^{(0)}w_z^{(1)} + v^{(1)}w_y^{(1)} - p_z^{(1)}/\rho = 0, \tag{10.33}$$

$$w_y^{(1)} = v_z^{(1)}, \quad u_y^{(1)} = w_x^{(1)}, \quad v_x^{(1)} = u_y^{(1)},$$

$$u^{(0)}\eta_x^{(1)} + u^{(1)}\eta_x^{(0)} + w^{(0)}\eta_z^{(1)} + w^{(1)}\eta_z^{(0)} + \eta_t^{(1)} - \eta^{(1)}v_y^{(1)} - v^{(2)} = 0 \quad \text{for } y = \eta^{(0)}(x,z,t),$$

$$u_1^{(1)}\ell_x + w_1^{(1)}\ell_z - v_1^{(2)} = 0 \quad \text{for } y = \ell(x,z),$$

$$p_2^{(1)} - p_1^{(1)} + \eta^{(1)}(p_{2y}^{(0)} - p_{1y}^{(0)}) = 0 \quad \text{for } y = \eta^{(0)}(x,z,t).$$

Some relations can be derived immediately from these equations. For the sake of brevity we introduce the following functions:

$$A_i(x,z,t) = u_{ix}^{(0)} + w_{iz}^{(0)}, \quad C_i(x,z,t) = (u_i^{(0)}\eta^{(0)})_x + (w_i^{(0)}\eta^{(0)})_z, \quad i=1,2,$$

$$B_1(x,z,t) = (u_1^{(0)}\ell)_x + (w_1^{(0)}\ell)_z, \quad B_2 = C_2 - C_1 + B_1.$$

Equations (10.28) may then be written

$$v_i^{(1)} = -yA_i + B_i, \quad i = 1,2. \tag{10.28'}$$

Then the fifth, first, and third equations of (10.33) give

$$
\begin{aligned}
u^{(1)} &= -\tfrac{1}{2} y^2 A_x + y B_x + h(x,z,t),\\
w^{(1)} &= -\tfrac{1}{2} y^2 A_z + y B_z + k(x,z,t),\\
h_z &= k_x,\\
v^{(2)} &= \tfrac{1}{6} y^3 (A_{xx} + A_{zz}) - \tfrac{1}{2} y^2 (B_{xx} + B_{zz}) - y(h_x + k_z) + \ell(x,z,t),\\
p^{(1)}/\rho &= \tfrac{1}{2} y^2 [A^2 + u^{(0)} A_x + w^{(0)} A_z + A_t] + y[AB + u^{(0)} B_x + w^{(0)} B_z + B_t] + g(x,z,t),
\end{aligned}
\qquad (10.34)
$$

where we have suppressed the subscripts indicating the fluid. The rest of the equations and the boundary conditions are still available to determine the unknown functions. We carry this out only for the case the upper fluid is missing. Then the last condition in (10.32) becomes $p^{(1)}/\rho\,(x,\eta^{(0)},z,t) = g\eta^{(1)}$ which allows one to determine $g(x,z,t)$ after $\eta^{(1)}$ is found. The next-to-the-last condition in (10.32) determines $\ell(x,z,t)$. The equations for h, k and $\eta^{(1)}$ are

$$
\begin{aligned}
u^{(0)} h_x + w^{(0)} h_z + h_t + u_x^{(0)} h + u_z^{(0)} k &= -g_x - B B_x,\\
u^{(0)} k_x + w^{(0)} k_z + k_t + w_x^{(0)} h + w_z^{(0)} k &= -g_z - B B_z,\\
u^{(0)} \eta_x^{(1)} + w^{(0)} \eta_z^{(1)} + \eta_t^{(1)} + A\eta^{(1)} &= [v^{(2)} - u^{(1)} \eta_x^{(0)} - w^{(1)} \eta_z^{(0)}]_{y=\eta^{(0)}},\\
h_z &= k_x,
\end{aligned}
\qquad (10.35)
$$

where

$$
g(x,z,t) = g\eta^{(1)} - \tfrac{1}{2}\eta^{(0)2}[A^2 + u^{(0)} A_x + w^{(0)} A_z + A_t] - \eta^{(0)}[AB + u^{(0)} B_x + w^{(0)} B_z + B_t],
$$

$$
\ell(x,z,t) = [u^{(1)} b_x + w^{(1)} b_z]_{y=b} - \tfrac{1}{6}\eta^{(0)3}[A_{xx} + A_{zz}] + \tfrac{1}{2}\eta^{(0)2}[B_{xx} + B_{zz}] - \eta^{(0)}(h_x + k_z).
$$

The solutions to these equations give the second-order corrections to the first-order shallow-water theory.

The equations resulting from the coefficients of ε^3 have been given by Keller [1948] for two dimensions, but will not be reproduced here.

The equations (10.32) for the first-order theory are nonlinear. In the theory of tides and seiches it is customary to simplify further by linearizing them in a manner similar to that used in deriving the equations for the infinitesimal-wave theory. Let $y = h$ be the surface of the undisturbed water and assume that one may make further expansions in a small parameter α: $u^{(0)} = \alpha u^{(01)} + \cdots$, $w^{(0)} = \alpha w^{(01)} + \cdots$, $\eta^{(0)} = h + \alpha \eta^{(01)} + \cdots$. After some easy manipulations one finds for the linearized approximation to (10.32) the equations

$$u_t^{(01)} + g\eta_x^{(01)} = 0, \quad w_t^{(01)} + g\eta_z^{(01)} = 0,$$

$$\eta_{tt}^{(01)} - g[\eta_x^{(01)}(h-b)]_x - g[\eta_z^{(01)}(h-b)]_z = 0. \tag{10.36}$$

If the bottom is flat, the equation for $\eta^{(01)}$ becomes the simple wave equation.

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D. Theory of Infinitesimal Waves

This chapter will deal with special solutions of the linearized equations derived in section 10α. This approximate theory has been very fruitful in its application to problems with various boundary configurations: the linear character of both the equations and boundary conditions allows one to use easily found simple solutions to construct other solutions satisfying special boundary conditions. The derivation of the equations in section 10α suggests the limitations of their use in physical problems: If L and V are a typical length and velocity associated with the physical problem, then, when the perturbation parameter ε is small, the surface elevation and velocities (or their deviation from a uniform flow) should be small with respect to L and V respectively. The smallness may not be uniform, but the quantities in question should approach zero point-wise with ε except at singular points.

11. The fundamental equations.

With few exceptions, this chapter will be concerned with the solution of a problem in potential theory. Let the (x, z)-plane be at the undisturbed free surface. We shall be seeking a function $\Phi(x,y,z,t)$, the velocity potential of the motion, satisfying the conditions (cf. 10.10)

$$\Delta\Phi = \Phi_{xx} + \Phi_{yy} + \Phi_{zz} = 0, \tag{11.1}$$

$$\Phi_{tt}(x,0,z,t) + g\Phi_y(x,0,z,t) = -\rho^{-1}\bar{p}_t(x,z,t),$$

$\Phi_n = V_n$ on solid boundaries,

where $\Delta\phi = 0$ is to be satisfied at all nonsingular points of the fluid in the region $y < 0$ and V_n is the normal velocity of the solid boundary at a given point. $\bar{p}(x,z,t)$ is a given pressure distribution on the free surface; in many problems it will be 0. The form of the free surface is given by:

$$\eta(x,z,t) = -\frac{1}{g}\Phi_t(x,0,z,t) - \frac{1}{\rho g}\bar{p}(x,z,t). \tag{11.2}$$

Two special cases occur frequently. If the motion is steady in a coordinate system moving with constant velocity c in the x-direction, then with x,y,z as moving coordinates, the free-surface boundary condition and equation of the surface are given by (cf. 10.15)

$$\begin{aligned} \varphi_y(x,0,z) + \frac{c^2}{g}\varphi_{xx}(x,0,z) &= \frac{c}{\rho g}\bar{p}_x(x,z), \\ \eta(x,z) &= \frac{c}{g}\varphi_x(x,0,z) - \frac{1}{\rho g}\bar{p}(x,z). \end{aligned} \tag{11.3}$$

If Φ and $\bar{p}$ are harmonic functions of the time, i.e.

$$\begin{aligned} \Phi(x,y,z,t) &= \varphi_1(x,y,z)\cos\sigma t + \varphi_2(x,y,z)\sin\sigma t \\ &= \mathrm{Re}\,\varphi(x,y,z)e^{-i\sigma t}, \end{aligned}$$

where

$$\varphi(x,y,z) = \varphi_1(x,y,z) + i\,\varphi_2(x,y,z)$$

and similarly for $\bar{p}$, then the free-surface condition and equation of the surface become

$$\begin{aligned} \varphi_{1y}(x,0,z) - \frac{\sigma^2}{g}\varphi_1(x,0,z) &= -\frac{\sigma}{\rho g}\bar{p}_2(x,z), \\ \varphi_{2y}(x,0,z) - \frac{\sigma^2}{g}\varphi_2(x,0,z) &= \frac{\sigma}{\rho g}\bar{p}_1(x,z), \\ \eta(x,z,t) = \frac{\sigma}{g}\left[\varphi_1(x,0,z)\sin\sigma t - \varphi_2(x,0,z)\cos\sigma t\right] & \\ - \frac{1}{\rho g}\left[\bar{p}_1(x,z)\cos\sigma t + \bar{p}_2(x,z)\sin\sigma t\right]. & \end{aligned} \tag{11.4}$$

In the few cases where we consider superposed fluids, viscous fluids or surface tension, we shall refer back to section 10 for the equations.

Use of complex variables. For two-dimensional irrotational motion, it is frequently advantageous to use complex variables. Let

$$z = x + iy, \quad f(z,t) = \Phi(x,y,t) + i\Psi(x,y,t)$$

where Φ and Ψ are velocity potential and stream function, respectively. (It should be clear from context whether z is being used for $x+iy$ or one of the horizontal coordinates.) Since the equations relating Φ and Ψ,

$$\Phi_x = \Psi_y, \quad \Phi_y = -\Psi_x,$$

are just the Cauchy-Riemann equations, the function $f(z,t)$ is an analytic function of z for all points z for which Φ_x and Φ_y exist. $f(z,t)$ will be called the "complex potential". The "complex velocity" is given by

$$w(z,t) = f'(z,t) = u - iv.$$

The boundary condition at the free surface in (11.1) can be expressed in the following equation in $f(z,t)$:

$$\mathrm{Re}\left\{ig f'(z,t) + \frac{d^2}{dt^2} f(z,t)\right\} = -\frac{1}{\rho} p_t(x,t) \text{ for } y=0. \tag{11.5}$$

The first equation of (11.3) becomes

$$\mathrm{Re}\left\{ig f'(z) + c^2 f''(z)\right\} = \frac{c}{\rho} p'(x) \text{ for } y=0. \tag{11.6}$$

However, equation (10.19) shows that this may also be taken in the form

$$\mathrm{Re}\left\{ig f(z) + c^2 f'(z)\right\} = \frac{c}{\rho} p(x) \text{ for } y=0.$$

If one may express $f(z,t) = f_1(z)\cos\sigma t + f_2(z)\sin\sigma t$, then the first of equations (11.4) becomes

$$\mathrm{Re}\{g f_k'(z) - \sigma^2 f_k(z)\} = (-1)^k \frac{\sigma}{\rho} P_{k-1,k}(x) \qquad (11.7)$$

$$k = 1, 2 .$$

We note that in order to express $f(z,t)$ in a manner analogous to that used for ϕ immediately preceding (11.4) one must introduce a second complex unit j which does not "interact" with i. Thus let $f(z) = f_1(z) + j f_2(z)$. Then $f(z,t) = \mathrm{Re}_j\, f(z) e^{-j\sigma t}$.

If $f(z)$ is an analytic function satisfying any one of the conditions (11.5)-(11.7), then $f^{(n)}(z)$ will also satisfy it.

12. Other boundary conditions.

The boundary conditions given in section 11 will not ordinarily be sufficient to ensure a unique solution to the problems in which the fluid occupies an unbounded region. An additional condition at infinity must be imposed upon the potential function. In certain cases the proper additional condition is fairly clear from the physical problem. For example, for a body moving steadily in an infinite ocean undisturbed except for the body, it seems reasonable to impose the condition that the fluid motion vanish far ahead of and far below the body. For the fluid motion produced by a stationary but steadily oscillating body, it seems reasonable to impose vanishing of the motion far below the body, but outgoing waves at infinity on all sides, if the body does not extend to infinity in some horizontal direction, the so-called "radiation condition".

If the body is not bounded in a horizontal direction, one may easily see that the radiation condition stated above cannot

be expected to be satisfied For example, suppose that waves are being generated by some type of oscillation of a vertical half-plane, say $z = 0$, $x > 0$, in which the oscillatory motion of the half-plane is independent of x. Then one will expect the generated waves to behave like outgoing plane waves from the two sides of the plane as $x \to \infty$: these will not satisfy the radiation condition in the direction Ox. On the other hand, one might expect that the influence of the edge at $x = 0$ would show up as waves satisfying the radiation condition. The formulation of proper boundary conditions in situations of this sort has been discussed by Peters and Stoker [1954]; see also Stoker, [1956; 1957, p. 109ff.].

In diffraction problems one customarily prescribes the form of an incoming wave and then seeks the scattered wave. The preceding remarks concerning the boundary conditions for waves generated by an oscillating body apply also to the scattered wave.

In more complicated physical situations it is not always clear what boundary conditions should be imposed at infinity, and errors have been made. For example, for a body which is both oscillating with a fixed frequency σ and moving with a steady average velocity c, one might reasonably expect no motion far ahead if c is large, but a radiation condition if c is small. However, the formulation of the boundary condition cannot be completed until the problem is partly solved.

The proper formulation of the boundary conditions at infinity can frequently be obtained by a method recommended by

Havelock [1917, 1949a] and used also by Brard [1948a, b], Stoker [1953, 1954], Stoker and Peters [1957], De Prima and Wu [1957], Wu [1957] and others. It consists in formulating an initial-value problem for which the desired steady-state problem is the limit as $t \to \infty$. For the initial-value problem the boundary condition at infinity is that the fluid motion vanishes everywhere.

13. Some mathematical solutions.

Some of the mathematical solutions to be derived in this section will provide solutions, without further modification, to certain physical problems, others, although apparently not acceptable physically, will provide fundamental solutions which can be used in constructing solutions to other more complicated physical problems. In all cases the fluid is assumed unbounded in a horizontal direction and either infinitely deep or with a horizontal bottom $y = -h$; the pressure on the free surface is taken to be zero everywhere. The solutions without singularities are obtained by the method of separation of variables, and are all harmonic in t. It will not be necessary to carry along the subscripts of (11.4).

13.1. Separation of the y-variable.

Assume that one may express φ by

$$\varphi(x, y, z) = Y(y)\varphi(x, z).$$

Then $\Delta_3 \varphi = \varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0$ becomes, after separation,

$$\Delta_2 \varphi + A\varphi = 0, \quad Y'' - AY = 0.$$

The two cases $A = m^2 > 0$ and $A = -m^2 < 0$ lead to different solutions.

$A > 0$. In this case $\varphi(x,z)$ satisfies the wave equation

$$\Delta_2 \varphi + m^2 \varphi = 0$$

and Y is given by

$$Y = a e^{my} + b e^{-my}.$$

If the fluid is infinitely deep and $\varphi_y(x,y,z)$ is to remain bounded as $y \to -\infty$, one must have $b = 0$. Then condition (11.4) requires

$$m = \frac{\sigma^2}{g}$$

and $\varphi(x,y,z)$ is of the form

$$\varphi(x,y,z) = e^{\sigma^2 y/g} \varphi(x,z). \tag{13.1}$$

If the fluid is of finite depth h, the boundary condition $\varphi_y(x,-h,z) = 0$ requires Y to take the form

$$Y = a \cosh m(y+h)$$

and condition (11.4) becomes

$$m \tanh mh = \sigma^2/g$$

an equation with two real solutions, say $\pm m_0$. In this case, one has

$$\varphi(x,y,z) = \cosh m_0(y+h)\, \varphi(x,z). \tag{13.2}$$

We note that, if $h_1 < h_2$ then $\sigma^2/g < m_0^{(2)} < m_0^{(1)}$. Also $m_0/h^{1/2} \to \sigma/g^{1/2}$ as $h \to 0$ and $m_0 \to \sigma^2/g$ as $h \to \infty$.

$A < 0$. In this case $\varphi(x,z)$ satisfies

$$\Delta_2 \varphi - m^2 \varphi = 0$$

and Y is given by

$$Y = a\cos my + b\sin my.$$

Condition (11 4) restricts Y further to

$$Y = C\left(m\cos my + \frac{\sigma^2}{g}\sin my\right).$$

If the fluid is infinitely deep, requiring φ to remain bounded imposes no further restriction. If the fluid is of depth h, then $\varphi_y(x,-h,z)=0$ requires m to satisfy the equation

$$m\tan mh = -\sigma^2/g$$

an equation with an infinite number of real solutions, $\pm m_1$, $\pm m_2$, ... In this latter case one may conveniently take Y in the form

$$Y = C\cos m(y+h).$$

The roots m_k satisfy $\frac{1}{2}(2k-1)\pi/h < m_k < k\pi/h$. For fixed h, $m_k h \to k\pi$ as $k \to \infty$; for fixed k, $m_k h \to k\pi$ as $h \to 0$, and $m_k h \to \frac{1}{2}(2k-1)\pi$ as $h \to \infty$.

For these two cases one finds then for $\varphi(x,y,z)$ the forms:

infinite depth:

$$\varphi(x,y,z) = C\left(m\cos my + \frac{\sigma^2}{g}\sin my\right)\varphi(x,z); \tag{13.3}$$

finite depth:

$$\varphi(x,y,z) = C\cos m(y+h)\,\varphi(x,z). \tag{13.4}$$

13b. <u>Further separation of variables</u>.

We now assume $\varphi(x,z) = X(x)Z(z)$ and substitute in each of the two equations for φ given above.

$A > 0$. In this case substitution in $\Delta\varphi + m^2\varphi = 0$ gives

$$X'' + (m^2 - k^2)X = 0 \qquad Z'' + k^2 Z = 0$$

(The equations obtained by replacing k^2 by $-k^2$ will give the solution obtained below for $A < 0$, with x and z interchanged.) The solution for Z is

$$Z = f\cos kz + g\sin kz = B\cos(kz + \gamma).$$

The solution for X depends upon the sign of $m^2 - k^2$:

$k^2 < m^2$: $\qquad X = c\cos\sqrt{m^2-k^2}\,x + d\sin\sqrt{m^2-k^2}\,x;$

$k^2 = m^2$: $\qquad X = cx + d,$

$k^2 > m^2$: $\qquad X = ce^{x\sqrt{k^2-m^2}} + de^{-x\sqrt{k^2-m^2}}.$

$A < 0$. Substitution in $\Delta_2\varphi - m^2\varphi = 0$ gives

$$X'' - (k^2 + m^2)X = 0, \quad Z'' + k^2 Z = 0,$$

which gives Z as above and

$$X = ce^{x\sqrt{k^2+m^2}} + de^{-x\sqrt{k^2+m^2}}.$$

(Substituting $-k^2$ for k^2 would give the solutions corresponding to $A > 0$ with x and z interchanged.) We may accumulate the preceding results to obtain the following fundamental solutions:

for infinite depth:

$$e^{\nu y}\left(a\cos x\sqrt{\nu^2-k^2} + b\sin x\sqrt{\nu^2-k^2}\right)\cos(kz+\delta)\cos(\sigma t+\tau), \quad k^2 < \nu^2,$$

$$e^{\nu y}(ax+b)\cos(kz+\gamma)\cos(\sigma t+\tau), \quad k^2 = \nu^2, \qquad (13.5)$$

$$e^{\nu y}\left(a e^{x\sqrt{k^2-\nu^2}} + b e^{-x\sqrt{k^2-\nu^2}}\right)\cos(kz+s)\cos(\sigma t+\tau), \quad k^2>\nu^2,$$

$$(m\cos my + \nu\sin my)\left(a e^{x\sqrt{k^2+m^2}} + b e^{-x\sqrt{k^2+m^2}}\right)\cos(kz+s)\cos(\sigma t+\tau),$$

where $\nu = \sigma^2/g$;

for finite depth:

$$\cosh m_0(y+h)\left(a\cos x\sqrt{m_0^2-k^2} + b\sin x\sqrt{m_0^2-k^2}\right)\cos(kz+s)\cos(\sigma t+\tau), \quad k^2<m_0^2,$$

$$\cosh m_0(y+h)(ax+b)\cos(kz+s)\cos(\sigma t+\tau), \quad k^2=m_0^2,$$

$$\cosh m_0(y+h)\left(a e^{x\sqrt{k^2-m_0^2}} + b e^{-x\sqrt{k^2-m_0^2}}\right)\cos(kz+s)\cos(\sigma t+\tau), \quad k^2>m_0^2, \qquad (13.6)$$

$$\cos m_i(y+h)\left(a e^{x\sqrt{k^2+m_i^2}} + b e^{-x\sqrt{k^2+m_i^2}}\right)\cos(kz+s)\cos(\sigma t+\tau),$$

where $m_0\tanh m_0h - \sigma^2/g = 0$ and $m_i\tan m_ih + \sigma^2/g = 0$.

The corresponding solutions for two dimensions may be obtained by setting $k=0$ and deleting the second and third equations in each group.

For either set of solutions only the first in each is bounded for all values of the variables for which $y \le 0$ or $-h \le y \le 0$. For two-dimensional motion it has been shown by Weinstein [~~1927~~ 1949] that the only function harmonic in $-h<y<0$ and satisfying (11.4) and $\varphi_y(x,-h)=0$ for which both φ and φ_y are bounded in $-h \le y \le 0$ is $\varphi = A\cosh m_0(y+h)\sin(m_0x+\alpha)$. Keldysh [1935] and Stoker [1947, pp. 7-9] have proved a similar theorem for the lower half-plane: If φ and $\varphi_x^2+\varphi_y^2$ are bounded for $y \le 0$ as $x^2+y^2 \to \infty$, the only φ satisfying (11.4) and harmonic everywhere in the half-plane $y \le 0$ is $Ae^{ky}\sin(kx+\alpha)$. Weinstein's theorem has been generalized by John [1950, p. 59] to three dimensions: If $\varphi(x,y,z)$ satisfies (11.4), $\varphi_y(x,-h,z)=0$,

$$\lim_{R\to\infty} \varphi(R\cos\alpha, y, R\sin\alpha) R^{\frac{1}{2}} e^{-m_1 R} = 0$$

and is harmonic everywhere in $-h \le y \le 0$, then $\varphi(x, y, z)$ is of the form (13.2) with $\varphi(x, z)$ an everywhere regular solution of

$$\Delta_2 \varphi + k^2 \varphi = 0$$

The condition at infinity is necessary, as the solution derived below in (13.8), $\varphi = I_0(m_1 R)\cos m_1(y+h)$ shows. The corresponding theorem for infinite depth was proved by Kochin [1940].

The equations for $\varphi(x, z)$ may also be separated in polar coordinates (R, θ), $x = R\cos\alpha$, $z = R\sin\alpha$. We give only the solutions:

infinite depth:

$$e^{\nu y}\left[A J_n(\nu R) + B Y_n(\nu R)\right]\cos(n\alpha + \delta)\cos(\sigma t + \tau),$$

$$(m\cos my + \nu\sin my)\left[A I_n(mR) + B K_n(mR)\right]\cos(n\alpha + \delta)\cos(\sigma t + \tau), \qquad (13.7)$$

where $\nu = \sigma^2/g$ and n is an integer.

finite depth:

$$\cosh m_0(y+h)\left[A J_n(m_0 R) + B Y_n(m_0 R)\right]\cos(n\alpha + \delta)\cos(\sigma t + \tau),$$

$$\cos m_i(y+h)\left[A I_n(m_i R) + B K_n(m_i R)\right]\cos(n\alpha + \delta)\cos(\sigma t + \tau),\ i \ge 1, \qquad (13.8)$$

where $m_0 \tanh m_0 h - \sigma^2/g = 0$, $m_i \tan m_i h + \sigma^2/g = 0$ and n is an integer. Here J_n, Y_n, I_n, K_n are Bessel functions (we use Watson's notation). Y_n and K_n are both singular at $R = 0$

but approach zero as $R \to \infty$; J_n and I_n are both finite at $R = 0$; J_n approaches zero as $R \to \infty$, I_n increases exponentially.

13. Singular solutions

In this section we shall find solutions of the problems set in section 11 which have singularities of simple type at a single point. We shall indicate proofs only for the case of simple sources, i. e. singularities of the form $[(x-a)^2+(y-b)^2+(z-c)^2]^{-1/2}$ or $\log[(x-a)^2+(y-b)^2]^{-1/2}$. We shall consider first the case of a stationary source of pulsating strength, then the case of a moving source. Three-dimensional problems are treated first.

Source of pulsating strength in three dimensions.

Let (a, b, c) be in the lower half-space. We wish to find a function

$$\Phi(x,y,z,t) = \varphi_1(x,y,z)\cos\sigma t + \varphi_2(x,y,z)\sin\sigma t$$

defined for $y \le 0$ except at (a, b, c) and satisfying

1) $\Delta\varphi_i = 0$ except at (a, b, c), $i = 1, 2$,

2) $\varphi_{iy}(x,0,z) - \nu\varphi_i(x,0,z) = 0$, $i = 1, 2$, $\nu = \sigma^2/g$,

3) $\Phi(x,y,z,t) = r^{-1}\cos\sigma t + \Phi_0(x,y,z,t)$, (13.9)

where Φ_0 is harmonic in the whole region $y < 0$,

4) $\lim_{y\to-\infty} \operatorname{grad}\varphi_i = 0$, $i = 1, 2$

5) $\lim_{R\to\infty} \sqrt{R}\left(\frac{\partial\varphi_1}{\partial R} + \nu\varphi_2\right) = 0$, $\lim_{R\to\infty} \sqrt{R}\left(\frac{\partial\varphi_2}{\partial R} - \nu\varphi_1\right) = 0$.

Here $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$ and $R^2 = (x-a)^2 + (z-c)^2$.
Condition 5), usually called the "radiation condition", requires the waves at infinity to be progressing outwards and imposes a uniqueness which would not otherwise be present. However, other such conditions could be imposed.

We assume that a solution Φ can be found in the form

$$\Phi(x,y,z,t) = [r^{-1} + \varphi_0(x,y,z)]\cos\sigma t + \varphi_2(x,y,z)\sin\sigma t. \tag{13.10}$$

φ_2 will be determined at the end so as to satisfy 5). Denote the double Fourier transform in x and z of φ by $\tilde{\varphi}$:

$$\varphi(x,y,z) = \frac{1}{2\pi}\int_0^{\infty}\int_{-\pi}^{\pi}\tilde{\varphi}(k,\vartheta,y)\,e^{ik(x\cos\vartheta + z\sin\vartheta)}\,d\vartheta\,dk$$

Then condition 1) applied to φ_0 becomes after transforming

$$\tilde{\varphi}_{0\,yy} - k^2\tilde{\varphi}_0 = 0$$

or

$$\tilde{\varphi}_0 = A_0(k,\vartheta)e^{yk} \tag{13.11}$$

where we have used 4) to discard the other solution.
From the known integral

$$(x^2+y^2+z^2)^{-\frac{1}{2}} = \frac{1}{2\pi}\int_0^{\infty}\int_{-\pi}^{\pi} e^{-k|y|}\,e^{ik(x\cos\vartheta + z\sin\vartheta)}\,d\vartheta\,dk \tag{13.12}$$

one may compute

$$\widetilde{r^{-1}} = e^{-k|y-b|} e^{-ik(a\cos\vartheta + c\sin\vartheta)} . \tag{13.13}$$

Substituting $\tilde{\varphi} + \widetilde{r^{-1}}$ in the transform of condition 2) gives

$$A(k,\vartheta) = \frac{k+\nu}{k-\nu} e^{kb} e^{-ik(a\cos\vartheta + c\sin\vartheta)} . \tag{13.14}$$

We now have, formally,

$$\varphi_2(x,y,z) = \frac{1}{2\pi} \int_0^\infty \int_{-\pi}^{\pi} \frac{k+\nu}{k-\nu} e^{k(y+b)} e^{ik[(x-a)\cos\vartheta + (z-c)\sin\vartheta]} \, d\vartheta \, dk .$$

Since the integrand has a singularity at $k = \nu$, the integral is not meaningful without further definition. We shall interpret the integral as a Cauchy principal value. Then

$$\begin{aligned} \varphi_1(x,y,z) &= \frac{1}{r} + \frac{1}{2\pi} PV \int_0^\infty \int_{-\pi}^{\pi} \frac{k+\nu}{k-\nu} e^{k(y+b)} e^{ik[(x-a)\cos\vartheta + (z-c)\sin\vartheta]} \, d\vartheta \, dk \\ &= \frac{1}{r} + \frac{1}{r_1} + \frac{\nu}{\pi} PV \int_0^\infty \int_{-\pi}^{\pi} \frac{1}{k-\nu} e^{k(y+b)} e^{ik[(x-a)\cos\vartheta + (z-c)\sin\vartheta]} \, d\vartheta \, dk \end{aligned} \tag{13.15}$$

where $r_1^2 = (x-a)^2 + (y+b)^2 + (z-c)^2$. The second equation may be derived easily from the first one by use of (13.12) suitably modified. φ_1 satisfies 1), 2) and 4); φ_2 is harmonic

in the whole region .

In order to satisfy 5) we shall first find the asymptotic form of φ_1 for large R. With polar coordinates

$$x - a = R\cos\alpha, \quad z - c = R\sin\alpha$$

one may write (13.15) as

$$\varphi_1(x,y,z) = \varphi_1(R,\alpha,y) = \frac{1}{r} + \frac{1}{r_1} + \frac{\nu}{\pi}\int_0^\infty\int_{-\pi}^{\pi}\frac{1}{k-\nu}e^{k(y+c)}e^{ikR\cos(\theta-\alpha)}\,d\theta\,dk$$

$$= \frac{1}{r} + \frac{1}{r_1} + \frac{4\nu}{\pi}\,PV\int_0^\infty\int_0^{\frac{1}{2}\pi}\frac{1}{k-\nu}e^{k(y+c)}\cos(kR\cos\omega)\,d\omega\,dk$$

$$= \frac{1}{r} + \frac{1}{r_1} + \frac{4\nu}{\pi}\,PV\int_0^\infty\int_0^1\frac{1}{\sqrt{1-\lambda^2}}\,\frac{1}{k-\nu}e^{k(y+c)}\cos R\lambda k\,d\lambda\,dk$$

$$= \frac{1}{r} + \frac{1}{r_1} + 2\nu\,PV\int_0^\infty\frac{1}{k-\nu}e^{k(y+c)}J_0(kR)\,dk$$

In the next to the last equation we shall change the order of integration, write

$$\cos R\lambda k = \cos R\lambda\nu\cos R\lambda(k-\nu) - \sin R\lambda\nu\sin R\lambda(k-\nu)$$

and use the following theorem from the theory of Fourier integrals [see, e.g., Bochner, Vorlesungen über Fouriersche Integrale, Leipzig, 1932, ch. I and § 8]: If $f(x)$ is a differentiable function in $[a,\infty]$, if $f''(x_0)$, $x_0 > a$, exists, and if $f(x)/x$ and $f'(x)/x$ are both absolutely integrable in $[a,\infty]$, then, as $R \to \infty$,

$$\int_a^\infty f(x)\,\frac{\sin R(x-x_0)}{x-x_0}\,dx = \pi f(x_0) + O\!\left(\frac{1}{R}\right), \quad PV\int_a^\infty f(x)\,\frac{\cos R(x-x_0)}{x-x_0}\,dx = O\!\left(\frac{1}{R}\right). \qquad (13.16)$$

Remembering that both r^{-1} and r_1^{-1} are $O(R^{-1})$, one finds

$$\varphi_1(x,y,z) = -4\nu e^{[illegible]} [illegible] \sin [illegible] + O(R^{-1})$$

The asymptotic expansion of this integral is well known [see, e.g. Erdelyi, Asymptotic expansions, Dover, New York, 1953, p. 48] and we may write

$$\varphi_1(x,y,z) = -2\pi\nu e^{\nu(y+b)} \sqrt{\frac{2}{\pi R\nu}} \sin\left(R\nu - \frac{\pi}{4}\right) + O\left(\frac{1}{R}\right).$$

If we can find a harmonic function φ_2 satisfying 1), 2) and 4) and having the asymptotic behavior

$$\varphi_2(x,y,z) = 2\pi\nu e^{\nu(y+b)} \sqrt{\frac{2}{\pi R\nu}} [illegible]\left([illegible] - \frac{\pi}{4}\right) + O\left(\frac{1}{R}\right)$$

then

$$\varphi_1(x,y,z) + \varphi_2(x,y,z) [illegible] =$$

$$-2\pi\nu e^{\nu(y+b)} \sqrt{\frac{2}{\pi R\nu}} \sin\left(R\nu - [illegible] - \frac{\pi}{4}\right) + O\left(\frac{1}{R}\right)$$

will be a solution. The following function fulfils the requirements:

$$\varphi_2(x,y,z) = 2\pi\nu e^{\nu(y+b)} J_0(R\nu)$$

[see the first equation (13.7) and Watson, Bessel functions, Cambridge, 1944, p. 199]. We note in passing that φ_1 has the same asymptotic behavior as

$$-2\pi\nu e^{\nu(y+b)} Y_0(R\nu).$$

The final result is

$$\Phi(x,y,z,t) = \left[\frac{1}{r} + PV\int_0^\infty \frac{k+\nu}{k-\nu} e^{k(y+h)} J_0(kR)\,dk\right]\cos\sigma t$$
$$+ 2\pi\nu e^{\nu(y+h)} J_0(\nu R)\sin\sigma t\,, \quad \nu = \sigma^2/g\,. \tag{13.17}$$

Haskind [1954], using a derivation having some similarity to that used below for the two-dimensional case, has found the following form for Φ:

$$\Phi(x,y,z,t) = \left[\frac{1}{r} + \frac{1}{r_1} - 2\nu e^{\nu y}\int_{-\infty}^{y} \frac{e^{-\nu y}}{r_1}\,dy - 2\pi\nu e^{\nu(y+h)} Y_0(\nu R)\right]\cos\sigma t$$
$$- 2\pi\nu e^{\nu(y+h)} J_0(\nu R)\sin\sigma t. \tag{13.17'}$$

It is sometimes convenient to use the complex form for the potential, $\varphi e^{-i\sigma t}$, with

$$\varphi(x,y,z) = \frac{1}{r} + PV\int_0^\infty \frac{k+\nu}{k-\nu} e^{k(y+h)} J_0(kR)\,dk + i\,2\pi\nu e^{\nu(y+h)} J_0(\nu R), \tag{13.17''}$$

for then $\operatorname{Re}\varphi e^{-i\sigma t}$ gives (13.17) and $\operatorname{Im}\varphi e^{-i\sigma t}$ the source potential for an outgoing wave with singularity of the form $r^{-1}\sin\sigma t$. Equation (13.17') may be written analogously. By deforming the path of integration in a familiar way one may also express $\varphi(x,y,z)$ in the form (cf. Havelock [1942, 1955]):

$$\varphi(x,y,z) = \frac{1}{r} + \frac{1}{r_1} - \frac{4\nu}{\pi}\int_0^\infty \left[\nu \cos k(y+b) - k \sin k(y+b)\right] \frac{K_0(kR)}{k^2+\nu^2}\, dk$$

$$- 2\pi\nu e^{\nu(y+b)} Y_0(\nu R) + i 2\pi\nu e^{\nu(y+b)} J_0(\nu R). \qquad (13.17'')$$

In the analogous problem for finite depth h one replaces 4) by 4′) $\varphi_y(x, -h, z) = 0$ and proceeds somewhat similarly. However, in order to satisfy 4′) it is convenient to look for a solution in the form

$$\Phi(x,y,z,t) = [r^{-1} + r_2^{-1} + \varphi_0(x,y,z)] \cos \sigma t + \varphi_2(x,y,z) \sin \sigma t,$$

where $r_2^2 = (x-a)^2 + (y+2h+b)^2 + (z-c)^2$. Equation (13.11) then becomes

$$\tilde{\varphi}_0 = A_0(k,\theta) \cosh k(y+h)$$

and (13.14), now more complicated because of r^{-1} and r_2^{-1}, becomes

$$A_0(k,\theta) = \frac{2(k+\nu)e^{-kh} \cosh k(b+h)}{k \sinh kh - \nu \cosh kh}\, e^{-ik(a\cos\theta + c\sin\theta)}.$$

The final formula for the velocity potential is

$$\Phi(x,y,z,t) = \left[\frac{1}{r} + \frac{1}{r_2} + \mathrm{P.V.}\int_0^\infty \frac{2(k+\nu)e^{-kh}\cosh k(b+h) \cosh k(y+h)}{k \sinh kh - \nu \cosh kh} J_0(kR)\, dk\right] \cos \sigma t$$

$$+ \frac{2\pi(m_0+\nu)e^{-m_0 h} \sinh m_0 h \cosh m_0(b+h) \cosh m_0(y+h)}{\nu h + \sinh^2 m_0 h} J_0(m_0 R) \sin \sigma t, \qquad (13.18)$$

where $m_0 \tanh m_0 h = \nu$, $\nu = \sigma^2/g$. The form of the last term of (13.18) may be altered by using the identities

$$\frac{e^{-m_0 h}\sinh m_0 h}{\nu h + \sinh^2 m_0 h} = \frac{2e^{-m_0 h}\cosh m_0 h}{2m_0 h + \sinh 2m_0 h} = \frac{m_0 - \nu}{m_0^2 h - \nu^2 h - \nu} .$$

John [1950, p. 95] has derived the following series for Φ, the analogue of (13.17):

$$\Phi(x,y,z,t) = 2\pi \frac{\nu^2 - m_0^2}{h m_0^2 - h\nu^2 - \nu} \cosh m_0(y+h) \cosh m_0 h \,[\,Y_0(m_0 R)\cos\sigma t - J_0(m_0 R)\sin\sigma t\,]$$

$$+ 4\sum_{k=1}^{\infty} \frac{m_k^2 + \nu^2}{h m_k^2 + h\nu^2 - \nu} \cos m_k(y+h) \cos m_k h \, K_0(m_k R)\cos\sigma t, \qquad (13.19)$$

where m_k, $k > 0$, are the positive real roots of $m \tan mh + \nu = 0$. Either expression may also be given in complex form as in (13.17′′).

Potential functions satisfying the condition (13. 9), but with $r^{-1}\cos\sigma t$ in 3) replaced by a higher-order singularity have been given by Thorne [1953] and Havelock [1955]. In fact, Thorne gives a rather complete census of the possible singular solutions for both two and three dimensions and for finite and infinite depth. Included are series expansions as well as integrals. For infinite depth the general expression which includes (13.17) is

$$\Phi(x,y,z,t) = \left[\frac{P_n^m(\cos\theta)}{r^{n+1}} + \frac{(-1)^n}{(n-m)!}\,PV\!\int_0^\infty \frac{k+\nu}{k-\nu}\,k^n e^{k(y+h)} J_m(kR)\,dk\right]\cos m\alpha \cos\sigma t$$

$$+ \frac{(-1)^m}{(n-m)!}\,2\pi\nu^{n+1} e^{\nu(y+h)} J_m(\nu R)\cos m\alpha \sin\sigma t, \qquad (13.20)$$

where $\cos\theta = (y-b)/r$, $x = R\cos\alpha$, $z = R\sin\alpha$. Here P_n^m are the associated Legendre polynominals defined by

$$P_n^m(\mu) = (1-\mu^2)^{m/2} \frac{d^m}{d\mu^m} P_n(\mu), \quad m \le n.$$

The asymptotic behavior of (13.20) is given by

$$\Phi(x,y,z,t) = \frac{(-1)^{m+1}}{(n-m)!} (2\nu)^{n+1} e^{\nu(y+b)} \sqrt{\frac{2}{\pi\nu R}} \sin\left(\nu R - \sigma t - \frac{2m+1}{4}\pi\right) + O\left(\frac{1}{R}\right).$$

It has been pointed out by both Havelock [1955] and MacCamy [1954] that solutions can be constructed which vanish much faster than this at infinity. Let the function of (13.20) be denoted by Φ_n. Then $\Phi_{n+1} - \nu(n-m+1)^{-1}\Phi_n$ is the following function:

$$\left[\frac{P_{n+1}^m(\cos\theta)}{r^{n+2}} - \frac{\nu}{n-m+1}\frac{P_n^m(\cos\theta)}{r^{n+1}} + (-1)^{n}\frac{P_{n+1}^m(-\cos\theta_1)}{r_1^{n+2}} + (-1)^{n}\frac{\nu}{n-m+1}\frac{P_n^m(-\cos\theta_1)}{r_1^{n+1}}\right]\cos mx \cos\sigma t, \qquad (13.21)$$

where $\cos\theta_1 = (y+b)/r_1$, $r_1^2 = (x-a)^2 + (y+b)^2 + (z-c)^2$. For $y=0$ and large R these solutions are $O(R^{-n-1})$ if m and n are both odd, $O(R^{-n-2})$ if one is even and one odd, and $O(R^{-n-3})$ if both are even. Although they have the form of standing waves, they satisfy the radiation condition because they decrease so rapidly with large R.

In addition to the papers cited above, one can find treatments of the submerged source of pulsating strength in Kochin [1940], Havelock [1942], John [1950, p. 92 ff.], where a

detailed discussion is given for the case of finite depth, Haskind [1944], and Liu [1952] The definition of the improper integral in (13 15) and following is not always the same in these different treatments In some cases the variable k is treated as complex and the path of integration deflected around the singularity $k = \nu$ by following a small semi-circle in the lower half of the k-plane The radiation condition is then automatically satisfied if one writes Φ in the complex form $\varphi e^{-i\sigma t}$, $\varphi = \varphi_1 + i\varphi_2$ Other treatments achieve the same end by introducing a "fictitious viscosity" μ which has the effect of replacing the singularity at $k = \nu$ by one at $k = \nu - i\mu$ and thus placing the path of integration below the singularity. In the end one must find the limit of the solution as $\mu \to 0$ The fictitious viscosity has no relation to real viscosity and may be considered a mathematical device to enable one to interpret an improper integral in a suitable way (for the purpose it seems to be infallible.

<u>Source and vortex of pulsating strength in two dimensions.</u>

The two-dimensional problem can be formulated analogously to (13 9), and solutions found in a similar manner. The fundamental singularities will now be of the form $\log r \cos \sigma t$, $r^{-n} \cos n\theta \cos \sigma t$ and $r^{-n} \sin n\theta \cos \sigma t$, $n = 1, 2, \cdots$. The results are given in the paper of Thorne [1953] cited earlier. We shall follow a different method here in order to illustrate the use of complex variables to solve suc problems.

We shall consider simultaneously a source of strength Q and a vortex of intensity Γ at the point $c = a + ib$, $b < 0$.

In the notation used at the end of section 11, we shall be looking for a function $f(z, t)$ analytic in z and of the form

$$f(z,t) = \left[\frac{\Gamma + iQ}{2\pi i}\log(z-c) + f_0(z)\right]\cos\sigma t + f_2(z)\sin\sigma t, \; Im\, c < 0$$

$$= f_1(z)\cos\sigma t + f_2(z)\sin\sigma t, \tag{13.22}$$

where f_0 and f_2 have no singularities in the lower half-plane. In addition, f_1 and f_2 must each satisfy the free-surface condition (11.7) which we write

$$Im\left\{f_k'(x-i0) + i\nu f_k(x-i0)\right\} = 0, \; \nu = \sigma^2/g, \; k = 1, 2. \tag{13.23}$$

Condition 4) of (13.9) will be taken in the somewhat stronger form,

$$|f_k'| \le M \text{ for } |z| \ge m \text{ and } \lim_{z\to\infty} |f_k'| = 0, \tag{13.24}$$

where m and M are given constants.
The radiation condition becomes:

$$\lim_{x\to\pm\infty} Re\left\{f_1' \pm \nu f_2\right\} = 0, \; \lim_{x\to\pm\infty} Re\left\{f_2' \mp \nu f_1\right\} = 0. \tag{13.25}$$

Following a method apparently originally due to Levi-Civita [see Tonolo, 1913], but used frequently by Keldysh [1935], Kochin [e.g., 1939], Stoker [1947], Lewy [1946] and others, we introduce the functions

$$A_k(z) = f_k'(z) + i\nu f_k(z) \qquad (13.26)$$

Then (13.23) becomes

$$\mathrm{Im}\, A_k(x + i0) = 0, \quad k = 1, 2, \qquad (13.27)$$

and (13.22) becomes: the two functions

$$A_1(z) - f_1(z) - \text{[illegible]} \frac{1}{z-c} - \nu \frac{\Gamma + iQ}{2\pi} \text{[illegible]}$$

and $A_2(z)$ are both regular everywhere in the lower half-plane. A function $A(z)$ with $\mathrm{Im}\, A(x + i0) = 0$ may be continued into the upper half-plane by defining $A(x + iy) = \overline{A(x - iy)}$, the bar indicating complex conjugate. Since A_2 is regular in the lower half-plane, the extended function will be regular in the whole plane. In addition, one may derive easily from (13.24) that $|A_2(z)| < C|z| + D$ for sufficiently large $|z|$: then, from the regularity of A_2, such an inequality holds in the whole lower half-plane and hence in the whole plane after reflection. It then follows from a known generalization of Liouville's Theorem [see Carathéodory, Theory of functions of a complex variable, vol. I, § 168, Chelsea, New York, 1954] that $A_2(z) = az + b$, where a and b are constants. It follows from (13.27) that a and b are real. The differential equation

$$f_2'(z) + i\nu f_2(z) = az + b$$

has the solution

$$f_2(z) = Ce^{-i\nu z} - \frac{ia}{\nu} z - \frac{ib}{\nu} + \frac{a}{\nu^2}.$$

The condition $\lim_{y\to-\infty} |f_2'| = 0$ requires $a = 0$. Thus, finally

$$f_2(z) = C_2 e^{-i\nu z} + iB_2, \quad B_2 \text{ real.}$$

One may set $B_2 = 0$ without loss of generality. Incidentally this provides a proof of the theorem of Stoker and Keldysh mentioned earlier (shortly after 13.6))

The function $A_1(z)$, after extension into the upper half-plane, will consist of four singular terms plus a function regular in the whole complex plane, say $A_3(z)$:

$$A_1(z) = \frac{\Gamma + iQ}{2\pi i} \frac{1}{z-c} + \nu \frac{\Gamma + iQ}{2\pi} \log(z-c) - \frac{\Gamma - iQ}{2\pi i} \frac{1}{z-\bar{c}}$$
$$+ \nu \frac{\Gamma - iQ}{2\pi} \log(z-\bar{c}) + A_3(z).$$

Since A_1 satisfies (13.27), and the four singular terms taken together also have vanishing imaginary part for $y=0$, the same must hold for A_3. Hence A_3 must have the same form as A_2. Substituting the resulting expression for A_1 in (13.26), one has a differential equation for $f_1(z)$.

The solution is

$$f_1(z) = \frac{\Gamma + iQ}{2\pi i} \log(z-c) + \frac{\Gamma - iQ}{2\pi i} \log(z-\bar{c}) - \frac{\Gamma - iQ}{\pi i} e^{-i\nu z} \int_{\infty}^{z} \frac{e^{i\nu u}}{u-\bar{c}}\, du$$
$$+ C_1 e^{-i\nu z} + iB_1,$$

where $\bar{c}$ is real and the path of integration is in the lower half-plane. As in the case [illegible], we may set [illegible]. C_1 and C_2 must now be chosen to satisfy (13.25). Making use of

$$\int_{-\infty}^{\infty} \frac{e^{i\nu u}}{u-\bar{c}}\, du = 2\pi i\, e^{i\nu\bar{c}},$$

one can show that

$$f_1' + \nu f_2 = -i\nu C_1 e^{i\nu z} + \nu C_2 e^{-i\nu z} + O(z^{-1}) \quad \text{as } z \to +\infty,$$

$$f_1' - \nu f_2 = -2\pi [\text{illegible}]\, Q e^{-i\nu(z-\bar{c})} - i\nu C_1 e^{-i\nu z} - \nu C_2 e^{i\nu z} + O(z^{-1})$$

$$\text{as } z \to -\infty.$$

This gives

$$C_1 = -(\text{[illegible]})\, e^{i\nu\bar{c}}, \qquad C_2 = -i(\text{[illegible]})\, e^{i\nu\bar{c}}.$$

One may easily verify that this choice of C_1 and C_2 does produce outgoing waves.

If one makes the change of variable $\nu(u - z) = k(z-\bar{c})$ in the integral term in f_1 and deforms the resulting path to Ox, one finds

$$-e^{-i\nu z}\int_{\infty}^{z} \frac{e^{i\nu u}}{u-\bar{c}}\, du = P.V.\int_{0}^{\infty} \frac{e^{-ik(z-\bar{c})}}{k-\nu}\, dk + \pi i\, e^{-i\nu(z-\bar{c})}.$$

Substituting this in the expression for f_1, one finally obtains

$$f(z,t) = \left[\frac{\Gamma + iQ}{2\pi i} \log(z-c) + \frac{\Gamma - iQ}{2\pi i} \log(z-\bar{c}) + \frac{\Gamma - iQ}{\pi i} \, PV\int_0^\infty \frac{e^{-ik(z-\bar{c})}}{k-\nu} \, dk \right] \cos \sigma t$$

$$- i(\Gamma - iQ) e^{-i\nu(z-\bar{c})} \sin \sigma t \qquad (13.28)$$

Singularities of higher order may be found by differentiating (13.28) with respect to z. The expression for $f'(z,t)$ may be put into a somewhat different form by using

$$\frac{\Gamma - iQ}{2\pi i} \, \frac{1}{z-\bar{c}} = \frac{\Gamma - iQ}{2\pi} \int_0^\infty e^{-ik(z-\bar{c})} \, dk .$$

Then

$$f'(z,t) = \left[\frac{\Gamma + iQ}{2\pi i} \, \frac{1}{z-c} - \frac{\Gamma - iQ}{2\pi} \, PV\int_0^\infty \frac{k+\nu}{k-\nu} e^{-ik(z-\bar{c})} \, dk \right] \cos \sigma t$$

$$- \nu (\Gamma - iQ) e^{-i\nu(z-\bar{c})} \sin \sigma t . \qquad (13.29)$$

One may continue differentiating, using either form for $f'(z,t)$. Thus, from (13.29)

$$f^{(n)}(z,t) = \left[\frac{\Gamma + iQ}{2\pi i} \, \frac{(-1)^{n-1}(n-1)!}{(z-c)^n} - \frac{\Gamma - iQ}{2\pi} (-i)^{n-1} \, PV\int_0^\infty k^{n-1} \frac{k+\nu}{k-\nu} e^{-ik(z-\bar{c})} \, dk \right] \cos \sigma t$$

$$- \nu^n (-i)^{n-1} (\Gamma - iQ) e^{-i\nu(z-\bar{c})} \sin \sigma t . \qquad (13.30)$$

By setting $\Gamma = 0$, $z - c = re^{i(\frac{\pi}{2}-\theta)} = ire^{-i\theta}$ (rather than the conventional $re^{i\theta}$ in order to distinguish easily symmetrical from unsymmetrical solutions) and taking the appropriate real or imaginary part, one finds the following formulas

for $\quad \phi(x, y, t)$:

$$\Phi(x,y,t) = \left[\frac{Q}{2\pi}\log\frac{r}{r_1} - \frac{Q}{\pi}\,\mathrm{PV}\int_0^\infty \frac{e^{k(y+b)}\cos k(x-a)}{k-\nu}\,dk\right]\cos\sigma t$$
$$- Q\,e^{\nu(y+b)}\cos\nu(x-a)\sin\sigma t,$$

$$\Phi(x,y,t) = \left[\frac{Q}{2\pi}\frac{\cos n\theta}{r^n} - \frac{(-1)^{n-1}}{(n-1)!}\frac{Q}{2\pi}\int_0^\infty k^{n-1}\frac{k+\nu}{k-\nu}e^{k(y+b)}\cos k(x-a)\,dk\right]\cos\sigma t$$
$$- \frac{(-1)^{n-1}}{(n-1)!}Q\,\nu^n e^{\nu(y+b)}\cos\nu(x-a)\sin\sigma t, \qquad (13.31)$$

$$\Phi(x,y,t) = \left[\frac{Q}{2\pi}\frac{\sin n\theta}{r^n} + \frac{(-1)^{n-1}}{(n-1)!}\frac{Q}{2\pi}\,\mathrm{PV}\int_0^\infty k^{n-1}\frac{k+\nu}{k-\nu}e^{k(y+b)}\sin k(x-a)\,dk\right]\cos\sigma t$$
$$+ \frac{(-1)^{n-1}}{(n-1)!}Q\,\nu^n e^{\nu(y+b)}\sin\nu(x-a)\sin\sigma t.$$

In the formula for the logarithmic singularity r_1 may be eliminated and the coefficient of $\cos\sigma t$ written as [v. John, 1950, p. 100]:

$$\frac{Q}{2\pi}\log r + \frac{Q}{2\pi}\,\mathrm{PV}\int_0^\infty\left[\frac{k+\nu}{(k-\nu)k}e^{k(y+b)}\cos k(x-a) + \frac{1}{k}e^{-k}\right]dk.$$

For water of finite depth the method used above does not work as conveniently because of the difficulty of formulating the boundary condition on the bottom, $\operatorname{Im} f'(x-ih)=0$, in terms of the function $A(z)$. However, it can be done, yielding a differential-difference equation for $f(z)$ which can be solved by use of Laplace transforms [cf. Bochner, Vorlesungen über Fouriersche Integrale, Leipzig, 1932, pp. 167-168]. The method used for the three-dimensional problem can also be carried through (see Haskind [1942b], John [1950], and Thorne [1953]).

It is convenient to separate the vortex from the source. The resulting functions are as follows.

vortex:

$$f(z,t) = \left[\frac{\Gamma}{2\pi i} \log(z-c) - \frac{\Gamma}{2\pi i} \log(z-c_2) \right.$$

$$\left. - \frac{\Gamma}{\pi} \, PV \int_0^\infty \frac{k+\nu}{k} \, \frac{e^{-kh} \sinh k(h+b) \sin k(z-a+ih)}{k \sinh kh - \nu \cosh kh} \, dk \right] \cos \sigma t \tag{13.32}$$

$$- \Gamma \, \frac{\nu+m_0}{m_0} \, \frac{e^{-m_0 h} \sinh m_0 h \sinh m_0(h-b) \sin m_0(z-a+ih)}{\nu h + \sinh^2 m_0 h} \sin \sigma t;$$

source:

$$f(z,t) = \left[\frac{Q}{2\pi} \log(z-c) + \frac{Q}{2\pi} \log(z-c_2) - \frac{Q}{\pi} \log ih \right. \tag{13.33}$$

$$\left. - \frac{Q}{\pi} \, PV \int_0^\infty \left\{ \frac{k+\nu}{k} \, \frac{e^{-kh} \cosh k(h+b) \cos k(z-a+ih)}{k \sinh kh - \nu \cosh kh} + \frac{e^{-kh}}{k} \right\} dk \right] \cos \sigma t$$

$$- Q \, \frac{\nu+m_0}{m_0} \, \frac{e^{-m_0 h} \sinh m_0 h \cosh m_0(h+b) \cos m_0(z-a+ih)}{\nu h + \sinh^2 m_0 h} \sin \sigma t.$$

Here $c_2 = a - ib - 2ih$. The remark following (13.18) concerning the form of the last term of that formula applies also here. The real part of either of these gives the corresponding potential function.

For the source, the integral representation and the series representation analogous to (13.19) are:

$$\Phi(x,y,t) = \Big[\frac{Q}{2\pi} \log \frac{r}{h} + \frac{Q}{2\pi} \log \frac{r_2}{h}$$

$$- \frac{Q}{\pi} PV \int_0^{\infty} \Big\{ \frac{k+\nu}{k} \, \frac{e^{-kh} \cosh k(h+b) \cosh k(y+h) \cos k(x-a)}{k \sinh kh - \nu \cosh kh} - \frac{e^{-kh}}{k} \Big\} dk \Big] \cos \sigma t$$

$$- 2Q \frac{\nu + m_0}{m_0} \, \frac{e^{-m_0 h} \sinh m_0 h \cosh m_0(h+b) \cosh m_0(y+h) \cos m_0(x-a)}{\ldots} \sin \sigma t$$

(13.34)

$$= Q \frac{1}{m_0} \frac{m_0^2 - \nu^2}{h m_0^2 - h\nu^2 + \nu} \cosh m_0(y+h) \cosh m_0(b+h) \sin[m_0|x-a| - \sigma t]$$

$$- Q \sum_{k=1}^{\infty} \frac{1}{m_k} \frac{m_k^2 + \nu^2}{h m_k^2 + h\nu^2 - \nu} \cos m_k(y+h) \cos m_k(b+h) e^{-m_k|x-a|} \sin \sigma t .$$

Thorne [1953] gives the potential functions for the higher-order singularities and the function for the logarithmic singularity in a form involving r and r_1 and hence more analogous to the one in (13.31). Voitsenya [1958] has derived the complex potential for a source-vortex situated in an infinitely deep fluid of density ρ_1 lying beneath another of density $\rho_2 < \rho_1$ and of thickness d.

Source of constant strength in uniform motion: three dimensions.

We shall assume the source moving in the direction Ox with constant velocity u_0. Let (x, y, z) be coordinates in a system moving with velocity u_0 in direction Ox and let the source be at (a, b, c), $b < 0$. Then, from section 11, we wish to find a function $\varphi(x, y, z)$ satisfying

1) $\Delta \varphi = 0$ except at (a, b, c),

2) $\varphi_{xx}(x, 0, z) + \nu \varphi_y(x, 0, z) = 0$, $\nu = g/u_0^2$,

(13.35)

3) $\varphi(x,y,z) = r^{-1} + \varphi_0(x,y,z)$

where φ_0 is harmonic in the region $y < 0$,

4) $\lim\limits_{y \to -\infty} \operatorname{grad} \varphi = 0$,

5) $\lim\limits_{x \to +\infty} \operatorname{grad} \varphi = 0$.

For fluid of finite depth h, 4) is replaced by 4′) $\varphi_y(x,-h,z) = 0$. Without condition 5), demanding vanishing of the motion far ahead of the source, the solution would not be unique. The profile of the free surface is obtained from $\eta(x,z) = u_0 g^{-1} \varphi_x(x,0,z)$. Strictly speaking, the solution of (13.35) will represent a sink, i.e. a source of strength -1. However, we shall continue to call such solutions sources.

A solution to this problem may be obtained by methods very similar to those used for the source of pulsating strength. The details will not be repeated, but can be found in Havelock [1932], Sretenskii [1937], Kochin [1937], Lunde [1951], Peters and Stoker [1957], Timman and Vossers [1955] and elsewhere. The result is

$$\varphi(x,y,z) = \frac{1}{r} - \frac{1}{r_1} - \frac{4\nu}{\pi}\int_0^{\frac{1}{2}\pi} d\vartheta \int_0^{\infty} \frac{e^{k(y+b)} \cos[k(x-a)\cos\vartheta] \cos[k(z-c)\sin\vartheta]}{k\cos^2\vartheta - \nu}\, dk$$

$$- 4\nu \int_0^{\frac{1}{2}\pi} e^{\nu(y+b)\sec^2\vartheta} \sin[\nu(x-a)\sec\vartheta] \cos[\nu(z-c)\sin\vartheta\sec^2\vartheta] \sec^2\vartheta \, d\vartheta, \tag{13.36}$$

where $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$, $r_1^2 = (x-a)^2 + (y+2h+b)^2 + (z-c)^2$, $\nu = g/U_0^2$.

The potential functions for higher-order singularities are unwieldy and will not be given. The one corresponding to $r^{-n-1}P_n(\cos\theta)$ can be easily obtained by n-fold differentiation with respect to b, if one remembers that

$$\frac{P_n(\cos\theta)}{r^{n+1}} = \frac{(-1)^n}{n!}\frac{\partial^n}{\partial b^n}\frac{1}{r}.$$

The dipole with axis in the direction Ox is obtained by differentiating (13.36) with respect to a and will be used later.

The velocity potential for a source moving in fluid of finite depth has been calculated by Sretenskii [1937] and by Haskind [1945b]. The form given below is essentially that given by Lunde [1951]:

$$\begin{aligned}
\varphi(x,y,z) &= \frac{1}{r} + \frac{1}{r_1} \\
&- \frac{4}{\pi}\int_0^{\frac{1}{2}\pi} d\theta \,\mathrm{P.V.}\!\int_0^\infty \frac{e^{-kh}\cosh k(y+h)[\cosh k(b+h)(k\cos^2\theta+\nu)-\nu]}{k\cos^2\theta\cosh kh-\nu\sinh kh} \\
&\qquad \cos[k(x-a)\cos\theta]\cos[k(z-c)\sin\theta]\,dk \\
&- 4\int_{\theta_0}^{\frac{1}{2}\pi} \frac{e^{-k_0h}\operatorname{sech}k_0h\cosh k_0(y+h)[\cosh k_0(b+h)(k_0\cos^2\theta+\nu)-\nu]}{\cos^2\theta-\nu h\operatorname{sech}^2k_0h} \\
&\qquad \sin[k_0(x-a)\cos\theta]\cos[k_0(z-c)\sin\theta]\,d\theta,
\end{aligned} \tag{13.37}$$

where $k_0 = k_0(\vartheta)$ is the real positive root of

$$k_0 - \nu \sec^2\vartheta \tanh k_0 h = 0 \,, \quad \vartheta_0 < \vartheta < \tfrac{1}{2}\pi \,,$$

where $\vartheta_0 = \arccos\sqrt{\nu h}$ if $\nu h = gh/u_0^2 < 1$, $\vartheta_0 = 0$ if $\nu h \geq 1$. As before $r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$ and $r_2^2 = (x-a)^2 + (y+2h+b)^2 + (z-c)^2$. We note that $k_0(\vartheta) < \nu \sec^2\vartheta$, $k_0(\vartheta) \to 0$ as $\vartheta \to \vartheta_0$, $k_0(\vartheta)/\nu \sec^2\vartheta \to 1$ as $\vartheta \to \frac{1}{2}\pi$ and $k_0(\vartheta) \to \nu\sec^2\vartheta$ as $h \to \infty$. In the double integral the principal value is necessary only for $\vartheta_0 < \vartheta < \frac{1}{2}\pi$, for the singularity does not occur in the denominator for $0 \leq \vartheta < \vartheta_0$. The part of the double integral with $0 \leq \vartheta < \vartheta_0$ approaches zero as $x \to \pm\infty$, so that no correction is necessary in order to satisfy condition 5). This is the explanation of the lower limit ϑ_0 in the second integral. In this integral the denominator vanishes only at $\vartheta = \vartheta_0$. One may verify that the integral is convergent by noting that

$$k_0'(\vartheta) = \frac{k_0 \sin 2\vartheta}{\cos^2\vartheta - \nu h \sec^2 k_0 h}$$

and rewriting it as an integral with respect to k_0. When $h \to \infty$, (13.37) reduces to a form of (13.36) in which r_1 is absorbed into the double integral.

For the stationary pulsating source the asymptotic form of the velocity potential for large R was found in the course

of deriving the potential function. For the moving source of constant strength the asymptotic form is more difficult to compute. Since the form of the free surface, $\eta = g^{-1}U\varphi_x(x,0,z)$ is of principal physical interest, we shall discuss the asymptotic form of φ_x instead of φ.

Introduce cylindrical coordinates $x - a = R\cos\alpha$, $z - c = R\sin\alpha$ into the x derivative of (13.36):

$$\varphi_x(R,\alpha,y) = \frac{-R\cos\alpha}{[R^2+(y-b)^2]^{3/2}} + \frac{R\cos\alpha}{[R^2+(y+b)^2]^{3/2}}$$

$$+ \frac{2\nu}{\pi}\int_0^{\pi/2} \sec^2\vartheta\, d\vartheta\, PV\!\int_0^\infty e^{k(y+b)} \frac{\sin[kR\cos(\vartheta-\alpha)] + \sin[kR\cos(\vartheta+\alpha)]}{k - \nu\sec^2\vartheta}\, k\, dk \tag{13.38}$$

$$-2\nu^2\int_0^{\pi/2} e^{\nu(y+b)\sec^2\vartheta}\{\cos[\nu R\sec^2\vartheta\cos(\vartheta-\alpha)] - \cos[\nu R\sec^2\vartheta\cos(\vartheta+\alpha)]\}\sec^3\vartheta\, d\vartheta.$$

For large R the first two terms taken together are $O(R^{-3})$. Apply the theorem (13.16) to the integral with respect to k. This gives, after combining with the second integral,

$$\varphi_x(R,\alpha,y) = 2\nu^2\int_0^{\pi/2} \sec^3\vartheta\, e^{\nu(y+b)\sec^2\vartheta}\{\cos[\nu R\sec^2\vartheta\cos(\vartheta-\alpha)][-1+\mathrm{sgn}\cos(\vartheta-\alpha)]$$

$$+\cos[\nu R\sec^2\vartheta\cos(\vartheta+\alpha)][-1+\mathrm{sgn}\cos(\vartheta+\alpha)]\}\, d\vartheta + O(R^{-1}). \tag{13.39}$$

Since φ_x is symmetric in α, we consider only $0 \le \alpha \le \pi$. We have for $0 \le \alpha \le \frac{1}{2}\pi$

$$\varphi_x(R,\alpha,z) = -4\nu^2\int_{\frac{1}{2}\pi-\alpha}^{\frac{1}{2}\pi} \sec^3\vartheta\, e^{\nu(z+b)\sec^2\vartheta}\cos[\nu R\sec^2\vartheta\cos(\vartheta-\alpha)]\,d\vartheta + O(R^{-1})$$

and for $\frac{1}{2}\pi < \alpha \le \pi$ (13.40)

$$\varphi_x(R,\alpha,z) = -4\nu^2\int_0^{\frac{1}{2}\pi} \sec^3\vartheta\, e^{\nu(z-b)\sec^2\vartheta}\cos[\nu R\sec^2\vartheta\cos(\vartheta+\alpha)]\,d\vartheta$$

$$-4\nu^2\int_0^{\alpha-\frac{1}{2}\pi} \sec^3\vartheta\, e^{\nu(z+b)\sec^2\vartheta}\cos[\nu R\sec^2\vartheta\cos(\vartheta-\alpha)]\,d\vartheta + O(R^{-1}).$$

Consider the two integrals containing $\cos(\vartheta+\alpha)$ and let

$$\lambda = \sec^2\vartheta\cos(\vartheta+\alpha)$$

Then for $0 \le \alpha \le \frac{1}{2}\pi$ the integral becomes

$$-4\nu^2\int_0^{-\infty}\frac{2\,e^{\nu(z+b)\sec^2\vartheta}}{\sin(2\vartheta+\alpha)-3\sin\alpha}\cos\nu R\lambda\,d\lambda .$$

If $\frac{1}{2}\pi < \alpha < \pi$, the lower limit is $\cos\alpha$.
In either case one may show that the coefficient of $\cos\nu R\lambda$ is single-valued, continuous, absolutely integrable and monotonically decreasing as a function of λ. By integration

by parts one may then establish the following estimates as $R \to \infty$ [cf. S. Bochner, Vorlesungen über Fouriersche Integrale, Leipzig, 1932, § 3]:

for $0 \leq \alpha \leq \frac{1}{2}\pi$

$$O(R^{-2});$$

for $\frac{1}{2}\pi < \alpha < \pi$

$$-4\nu \frac{e^{\nu(y+b)}}{R\sin\alpha} \sin(\nu R\cos\alpha) + O(R^{-2}).$$

If $\alpha = \pi$, the two integrals in (13.40) combine to give

$$8\nu^2 \int_1^\infty e^{\nu(y+b)\lambda^2} \frac{\lambda^2}{\sqrt{\lambda^2-1}} \cos\nu R\lambda \, d\lambda =$$

$$4\nu^2\sqrt{2\pi} \; \frac{e^{\nu(y-b)}}{\sqrt{\nu R}} \cos(\nu R - \tfrac{3}{4}\pi) + O(R^{-1}).$$

Consider now the remaining integral in (13.40), and let

$$\mu(\vartheta) = \sec^2\vartheta \cos(\vartheta-\alpha).$$

The integral takes the form

$$-8\nu^2 \int_{\cos\alpha}^{0} e^{\nu(y+b)\sec^2\vartheta} \frac{\cos\nu R\mu}{\sin(2\vartheta-\alpha)+3\sin\alpha} \, d\mu.$$

The denominator now becomes zero when

$$\tan\vartheta = -\frac{1}{4}\cot\alpha\,[1 \pm \sqrt{1-8\tan^2\alpha}\,], \tag{13.41}$$

an equation which has real roots when $\tan^2\alpha \le 1/8$, i.e. when

$$180^\circ - 19^\circ 28' < \alpha < 180^\circ.$$

When $\frac{1}{2}\pi < \alpha < \pi - \mathrm{Arcsin}\,\frac{1}{3} = \alpha_c$, the Fourier-integral estimate used for the other integral may be applied to give

$$4\nu e^{\nu(y+b)}\,\frac{\sin(\nu R\cos\alpha)}{R\sin\alpha} + O(R^{-2}).$$

When $\alpha_c < \alpha < \pi$, ϑ is a two-branched function of μ and the resulting two integrals each have singularities at one of the limits. Thus the elementary method of analysis used above can no longer be applied. However, a modification of the method above can be carried through [see, e.g., Erdélyi, Asymptotic expansions, Dover, New York, 1956, pp. 46-56]; the classical treatment is by the method of stationary phase which is well discussed in Stoker [1957, ch. 8].

The estimates already derived are of the same order as the remainder term in (13.39). Analysis of this term produces terms which cancel the terms in $1/R\sin\alpha$ already derived, thus removing an apparent singular behavior near the x-axis.

The asymptotic form for the surface $\eta(R,\alpha)$ above a source of strength m (i.e. $-m/r$) is as follows:

for $0 \le \alpha < \pi - \mathrm{Arcsin}\,\frac{1}{3} = \alpha_c$ (13.42)

$$\eta(R,\alpha) = O\left((\nu R)^{-2}\right);$$

for $\alpha = \alpha_c$

$$\eta(R,\alpha_c) = 4\cdot 3^{5/6}\,\Gamma(\tfrac{1}{3})\,\frac{w\nu}{u_0}\,(\nu R)^{-1/3}\,e^{\frac{3}{2}\nu b}\cos\left(\tfrac{\sqrt{3}}{2}\nu R\right) + O\left((\nu R)^{-1}\right);$$

for $\alpha_c < \alpha < \pi$

$$\eta(R,\alpha) = 4\sqrt{2\pi}\,\frac{w\nu}{u_0}\,\frac{(\nu R)^{-1/2}}{[1-9\sin^2\alpha]^{1/4}}\left\{\sec^{3/2}\vartheta_1\, e^{\nu b\sec^2\vartheta_1}\cos\left(\nu R\mu_1 - \tfrac{1}{4}\pi\right)\right.$$

$$\left. + \sec^{3/2}\vartheta_2\, e^{\nu b\sec^2\vartheta_2}\cos\left(\nu R\mu_2 + \tfrac{1}{4}\pi\right)\right\} + O\left((\nu R)^{-1}\right);$$

for $\alpha = \pi$

$$\eta(R,\pi) = -4\sqrt{2\pi}\,\frac{w\nu}{u_0}\,(\nu R)^{-\frac{1}{2}}\,e^{\nu b}\cos\left(\nu R - \tfrac{3}{4}\pi\right) + O\left((\nu R)^{-1}\right).$$

Here ϑ_1 and $\vartheta_2 > \vartheta_1$ are the two roots (13.41) and $\mu_1 (<0)$ and $\mu_2 < \mu_1$ the corresponding values of $\sec^2\vartheta\cos(\vartheta-\alpha)$. As $\alpha \to \alpha_c$, $\vartheta_i \to \mathrm{Arctan}\,\frac{1}{2}\sqrt{2}$, $\mu_i \to -\frac{1}{2}\sqrt{3}$; as $\alpha \to \pi$, $\vartheta_1 \to 0$, $\mu_1 \to -1$, $\vartheta_2 \to \frac{1}{2}\pi$, $\mu_2 \to -\infty$.

In order to have some idea of the form of the free surface far behind the source, one may graph the curves

$$\nu R \mu_1(\alpha) - \frac{1}{4}\pi = -2n\pi\ , \quad \nu R \mu_2(\alpha) + \frac{1}{4}\pi = -2n\pi\ , \quad n > 0,$$

showing the traces of the wave crests in the region

$$\pi - \operatorname{Arcsin}\frac{1}{3} < \alpha < \pi + \operatorname{Arcsin}\frac{1}{3}\ .$$

This gives the well known pattern shown in Figure 1a. The first equation gives the transverse waves, the second one the diverging waves. The wave length along $\alpha = \pi$ is $2\pi/\nu$ and along the boundary lines $4\pi\sqrt{3}/3\nu$. The expansion is not suitable in the region near the boundary lines $\alpha = \alpha_c$. As $\alpha \to \alpha_c$, $\alpha > \alpha_c$, the term $[1 - 9\sin^2\alpha]^{-1/4} \to 0$ and the amplitudes become infinite. A special investigation of the region near $\alpha = \alpha_c$ is necessary and shows $(\nu R)^{-1/3}$ as leading term; η may be expressed in terms of Airy functions (cf. Ursell [1959]).

Essentially the same pattern is produced by a moving concentrated pressure on the free surface; it was first analysed by Kelvin [1906 = Papers, vol. IV, pp. 407-413]. The asymptotic behavior for moving pressure distributions has been extensively studied (e.g., Hogner [1923]; Teturô Inui [1936]; Peters [1949]; Bartels and Downing [1955]). Lamb [1926] has given the asymptotic form of the surface over a moving submerged dipole. The form of the surface near the moving dipole has been investigated by Havelock [1928], who gives traces of the profile on planes α = const. for several values of α between $\frac{1}{2}\pi$ and π (the radial lines of Figures 1b and 1c) and for $|b\nu| = 1/2$ and 4. Havelock's computations were later used

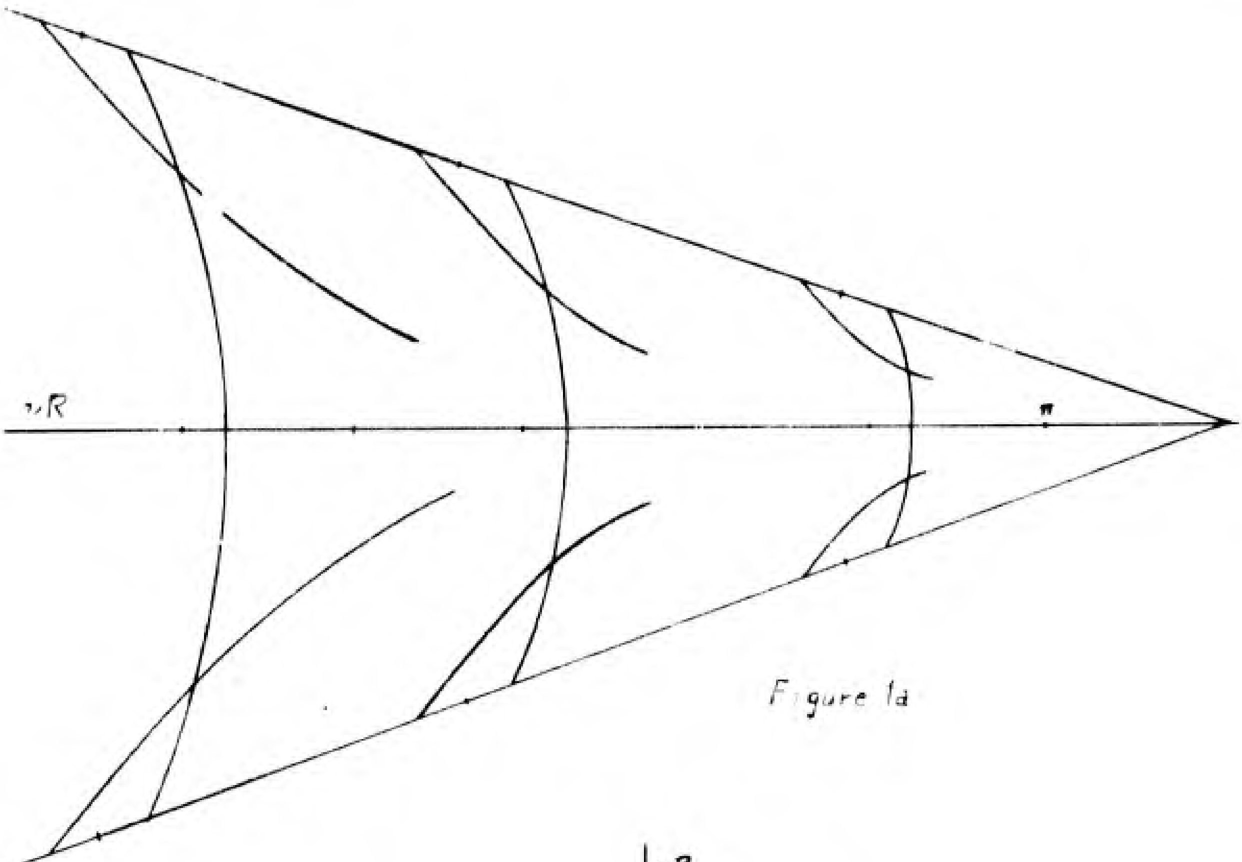

Figure 1a

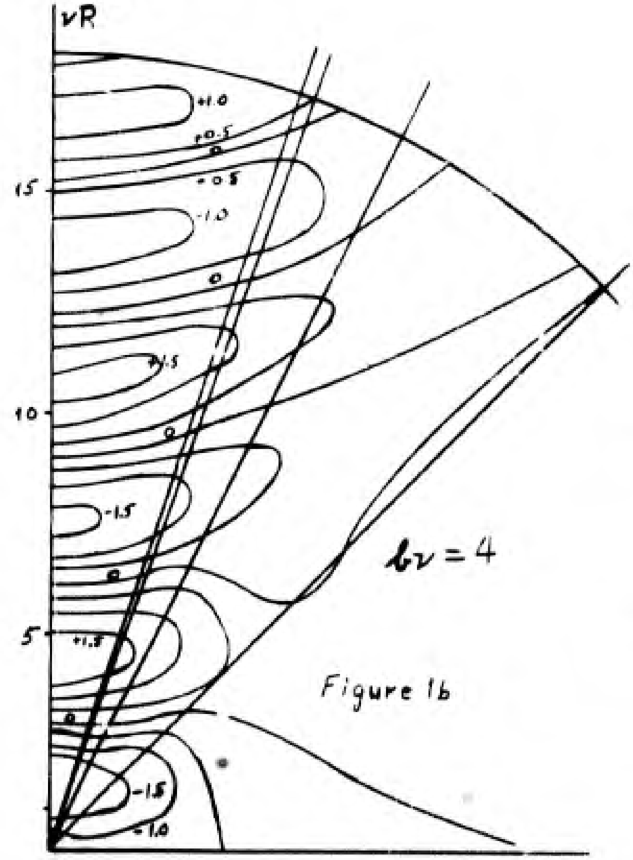

Figure 1b

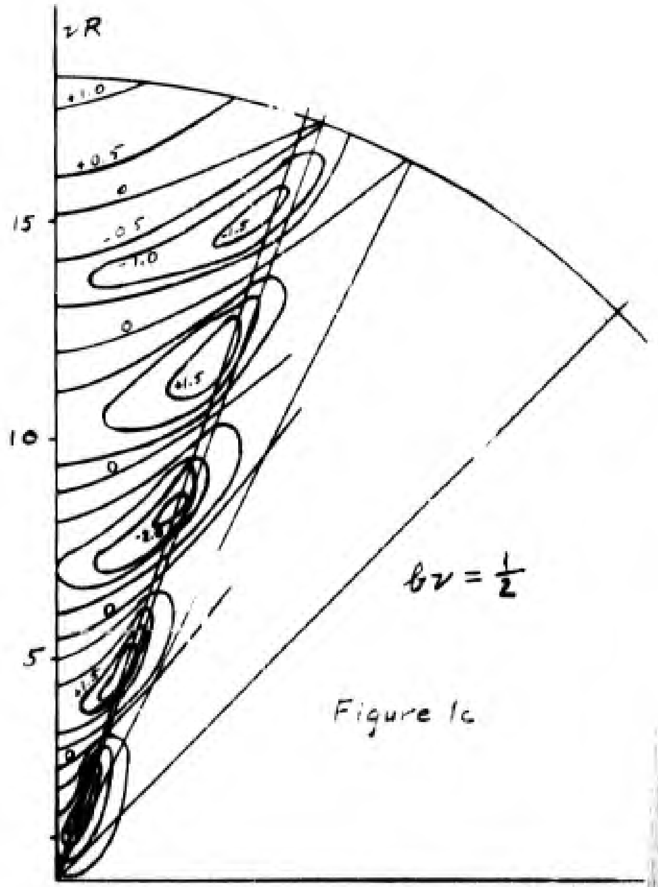

Figure 1c

by Wigley [1930] to produce the contour curves shown in Figures 1b and 1c.

A similar analysis may be made for (13.37), a source moving in fluid of finite depth. For a moving pressure distribution the problem has been treated by Havelock [1908] and Teturô Inui [1936]. The pattern is modified as follows. If $\nu h > 1$, the pattern is qualitatively like that for $h = \infty$. However, the wedge within which the disturbance is chiefly contained has a wider aperture and as $\nu h \to 1$ the aperture approaches $\frac{1}{2}\pi$ radians on each side of the line of motion. In addition, the wave length of the transverse wave system increases and approaches infinity as $\nu h \to 1$. When $\nu h \geq 1$, the transverse wave system is missing completely, but diverging waves still occur in a wedge of aperture varying from π to 0 as $\nu h \to 0$. (See also Ekman [1906], who has considered the free surface over a dipole on a flat bottom.)

Figure 2 from Havelock [1908] shows the half-angle of the aperture.

Kochin [1938c] has gone further in this type of problem. He has derived the potential for source situated in a fluid of density ρ_1 and depth h, bounded below by a plane, over which is lying another fluid of density $\rho_2 < \rho_1$, extending infinitely far upwards. The lower fluid moves with velocity c_1, the upper with velocity c_2 in the same direction. He also finds the asymptotic behavior of the solution.

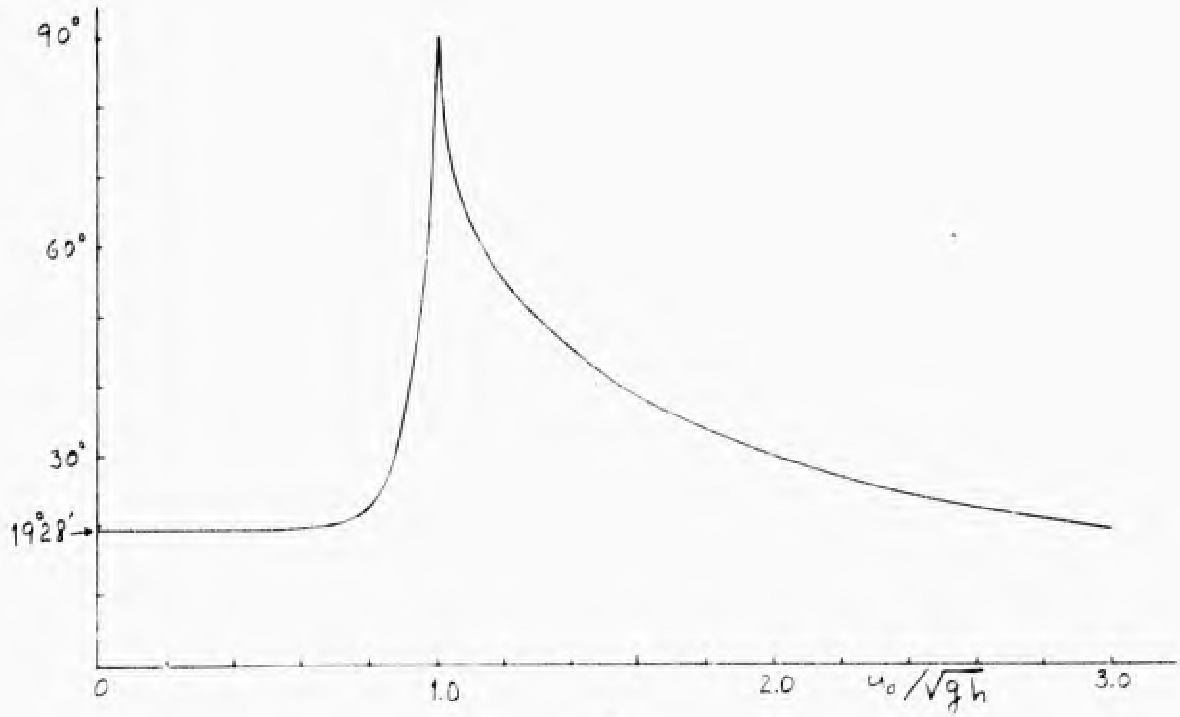

Figure 2

Singularities of constant strength in uniform motion: two dimensions.

For submerged sources and vortices in two-dimensional motion the complex-variables method used for the pulsating source may again be applied. For infinitely deep fluid, the computation has been carried out in this way by Keldysh and Lavrent'ev [1937] and Kochin [1937]; a detailed exposition is given in the textbook of Kochin, Kibel' and Roze [1948, ch. VIII, § 19]. Havelock [1927] and Sretenskii [1938] have treated the problem by different methods. The complex velocity potential for a combined source of strength Q and vortex of intensity Γ at $c = a + ib$, $b < 0$, is given by

$$f(z) = \frac{\Gamma + iQ}{2\pi i}\log(z-c) - \frac{\Gamma - iQ}{2\pi i}\log(z-\bar{c}) + \frac{\Gamma - iQ}{\pi i} e^{-i\nu z}\int_{\infty}^{z} \frac{e^{i\nu u}}{u-\bar{c}}\,du,$$

$$= \frac{\Gamma + iQ}{2\pi i}\log(z-c) - \frac{\Gamma - iQ}{2\pi i}\log(z-\bar{c}) - 2(\Gamma - iQ)e^{-i\nu(z-\bar{c})} + \frac{\Gamma - iQ}{\pi i} e^{-i\nu z}\int_{-\infty}^{z} \frac{e^{i\nu u}}{u-\bar{c}}\,du, \qquad (13.43)$$

$$= \frac{\Gamma + iQ}{2\pi i}\log(z-c) - \frac{\Gamma - iQ}{2\pi i}\log(z-\bar{c}) - \frac{\Gamma - iQ}{\pi i}\,\mathrm{PV}\int_{0}^{\infty} \frac{e^{-ik(z-\bar{c})}}{k-\nu}\,dk - (\Gamma - iQ)e^{-i\nu(z-\bar{c})}.$$

The real velocity potential for, say, a submerged source can be obtained from any of these equations. The last one gives a form analogous to (13.36):

$$\varphi(x,y) = \frac{Q}{2\pi}\log\frac{r}{c} + \frac{Q}{2\pi}\log\frac{r_1}{c} + \frac{Q}{\pi}\,\mathrm{PV}\int_{0}^{\infty} \frac{e^{k(y+b)}\cos k(x-a)}{k-\nu}\,dk + Qe^{\nu(y+b)}\sin\nu(x-a). \qquad (13.44)$$

Higher-order singularities can be obtained by differentiating (13.43). The complex velocity potential for a dipole of moment M and axis in the direction $e^{i\alpha}$ is given by

$$f(z) = -\frac{M}{2\pi}\frac{e^{i\alpha}}{z-c} + \frac{M}{2\pi}\frac{e^{-i\alpha}}{z-\bar{c}} - \frac{iM\nu}{\pi}e^{-i\alpha}e^{-i\nu z}\int_{\infty}^{z}\frac{e^{i\nu u}}{u-\bar{c}}\,du$$

$$= -\frac{M}{2\pi}\frac{e^{i\alpha}}{z-c} + \frac{M}{2\pi}\frac{e^{-i\alpha}}{z-\bar{c}} + \frac{iM\nu}{\pi}e^{-i\alpha}\,\mathrm{PV}\int_{0}^{\infty}\frac{e^{-ik(z-\bar{c})}}{k-\nu}\,dk - M\nu e^{-i\alpha}e^{-i\nu(z-\bar{c})}. \qquad (13.45)$$

If in the last term of the first equation of either (13.43) or (13.45), one makes use of the identity

$$e^{-i\nu z}\int_{\infty}^{z} \frac{e^{i\nu u}}{u-\bar{c}}\,du = \int_{-\infty}^{0} \frac{-e^{-i\nu u}}{z-u-\bar{c}}\,du,$$

it is not difficult to see that this last term is equivalent to a distribution of dipoles on the ray from $\bar{c}$ to $-\infty$ parallel to the x-axis. The moment density and axis can be determined for the three cases, source, vortex and dipole, by comparison of the integrand with the first term of (13.45).

For the case of finite depth the complex velocity potential has been calculated by Tikhonov [1940] and is also given by Haskind [1945a] for both source and vortex. We give separately source, vortex and dipole:

source:

$$\begin{aligned} f(z) = {} & \frac{Q}{2\pi}\log(z-c) + \frac{Q}{2\pi}\log(z-\bar{c}) \\ & + \frac{2Q}{\pi}\,i\nu\int_{0}^{\infty}\frac{k+\nu}{k}\,e^{-kh}\,\frac{\cosh k(\zeta+h)}{\nu\sinh kh - k\cosh kh}\,\sin^{2}\tfrac{1}{2}k(z-a+ih)\,dk \\ & - \frac{Q\nu}{k_0}\,\frac{\cosh k_0(\zeta+h)}{\nu h-\cosh^{2}k_0 h}\,\sin k_0(z-a+ih); \end{aligned} \qquad (13.46)$$

vortex:

$$f(z) = \frac{\Gamma}{2\pi i}\log(z-c) - \frac{\Gamma}{2\pi i}\log(z-c_2)$$

$$- \frac{\Gamma}{\pi}\, \mathrm{PV}\!\int_0^\infty \frac{k+\nu}{k}\, e^{-kh}\, \frac{\sinh k(b+h)}{\nu \sinh kh - k\cosh kh}\, \sin k(z-a+ih)\, dk$$

$$+ \frac{\Gamma\nu}{k_0}\, \frac{\sinh k_0(b+h)}{\nu h - \cosh^2 k_0 h}\, \cos k_0(z-a+ih) \,. \tag{13.47}$$

dipole:

$$f(z) = -\frac{M}{2\pi}\frac{e^{i\alpha}}{z-c} - \frac{M}{2\pi}\frac{e^{-i\alpha}}{z-c_2}$$

$$- \frac{M}{2\pi}\, \mathrm{PV}\!\int_0^\infty (k+\nu) e^{-kh}\, \frac{e^{i\alpha}\sin k(z-c) + e^{-i\alpha}\sin k(z-c_2)}{\nu\sinh kh - k\cosh kh}\, dk$$

$$+ \frac{\nu M}{2}\, \frac{e^{i\alpha}\cos k_0(z-c) + e^{-i\alpha}\cos k_0(z-c_2)}{\nu h - \cosh^2 k_0 h} \,. \tag{13.48}$$

Here $c_2 = a - ib + 2ih$ and the last summand in each of (13.46) to (13.48) is to be deleted if $\nu h = gh/c^2 \le 1$; k_0 is the positive real root of $\nu\sinh kh - k\cosh kh = 0$, which exists only if $\nu h > 1$.

Asymptotic form of these functions as $x \to -\infty$ is easily seen to be given by double the last term in each expression.

When $\nu h < 1$, the disturbance is only local, a fact which corresponds to the absence of transverse waves behind the three-dimensional source for $\nu h < 1$.

Kochin [1937a,b] has derived the complex velocity potential when fluid of density ρ_2 overlies the fluid of density $\rho_1 > \rho_2$ containing the singularity. The lower fluid may be of infinite or finite depth; the upper one is taken infinitely deep. Their velocities may be different.

Source of variable strength, starting from rest and following an arbitrary path.

Consider now a source whose position and strength at time $t \geq 0$ are given by $(a(t), b(t), c(t))$ and $m(t)$, where $b(t) < 0$. Let $m(t) = 0$ for $t < 0$. The conditions to be satisfied by the velocity potential $\Phi(x, y, z, t)$ are

1) $\Delta\Phi = 0, \quad y < 0, \quad (x, y, z) \neq (a(t), b(t), c(t)),$

2) $\Phi_{tt}(x, 0, z, t) + g\Phi_y(x, 0, z, t) = 0,$

3) $\Phi(x, y, z, t) = m(t) r^{-1} + \Phi_0(x, y, z, t)$, Φ_0 harmonic everywhere in $y < 0$,

4) $\lim_{y \to -\infty} \operatorname{grad} \Phi = 0$ for all x, z and t,

5) $\lim_{R \to \infty} \operatorname{grad} \Phi = 0$ for all t,

6) $\Phi(x, 0, z, 0) = \Phi_t(x, 0, z, 0) = 0.$

Here $r^2 = (x - a(t))^2 + (y - b(t))^2 + (z - c(t))^2$; $R^2 = (x - a(t))^2 + (z - c(t))^2$.

If one assumes a solution in the form

$$\Phi = mr^{-1} - mr_1^{-1} + \Phi_1$$

where $r_1^2 = (x-a)^2 + (y+b)^2 + (z-c)^2$, then Φ_1 must be harmonic in $y < 0$ and satisfy 4), 5), 6) and

$$\Phi_{1tt}(x,0,z,t) + g\Phi_{1y}(x,0,z,t) = -2g\, m(t)\, b(t) \left[(x-a)^2 + b^2 + (z-c)^2\right]^{-3/2}, \; t \geq 0.$$

It follows from the conditions that, for $t < 0$, Φ_1 = constant, which we may take as zero. Let $\bar{\Phi}_1$ be the Laplace transform of Φ_1:

$$\bar{\Phi}_1(x,y,z,s) = \int_0^{\infty} e^{-st}\, \Phi_1(x,y,z,t)\, dt.$$

Then $\bar{\Phi}_1$ is a harmonic function in $y < 0$ satisfying 4) and 5) for each s and also, after making use of 6), the condition

$$s^2 \bar{\Phi}_1(x,0,z,s) + g\bar{\Phi}_{1y}(x,0,z,s) = -2g\int_0^{\infty} e^{-st} m(t)\, b(t) \left[(x-a)^2 + b^2 + (z-c)^2\right]^{-3/2} dt.$$

Since

$$s^2 \bar{\Phi}_1(x,y,z,s) + g\bar{\Phi}_{1y}(x,y,z,s) + 2g\int_0^{\infty} e^{-st} m(t) (y + b(t)) \left[(x-a)^2 + (y+b)^2 + (z-c)^2\right]^{-3/2} dt$$

is a harmonic function in $y < 0$ vanishing on $y = 0$ and at infinity, it is identically zero. Making use of (13.12) differentiated with respect to y and changing the order of integration

one obtains

$$s^2 \bar{\Phi}_1(x,y,z,s) + g\bar{\Phi}_{1y}(x,y,z,s) =$$

$$= \frac{g}{\pi}\int_0^\infty k\,dk \int_0^\infty dt\, e^{-st} m(t) e^{k(y+b)} \int_{-\pi}^{\pi} d\vartheta\, e^{ik[(x-a)\cos\vartheta + (z-c)\sin\vartheta]}$$

$$= 2g\int_0^\infty k\,dk \int_0^\infty dt\, e^{-st} m(t) e^{k(y+b)} J_0\left(k[(x-a)^2+(z-c)^2]^{1/2}\right).$$

The solution for $\bar{\Phi}_1$ is

$$\bar{\Phi}_1(x,y,z,s) = 2g\int_0^\infty dk\, \frac{k}{s^2+gk}\int_0^\infty dt\, e^{-st} m(t) e^{k(y+b)} J_0(kR(t)).$$

Making use of the convolution theorem and the fact that $(s^2+gk)^{-1}$ is the transform of $(gk)^{-\frac{1}{2}} \sin (gk)^{\frac{1}{2}} t$, one may find the original function $\Phi_1(x,y,z,t)$:

$$\Phi_1(x,y,z,t) = 2\int_0^\infty dk\,(gk)^{\frac{1}{2}}\int_0^t d\tau\, \sin[(gk)^{\frac{1}{2}}(t-\tau)]\, m(\tau) e^{k(y+b(\tau))} J_0(kR(\tau))$$

For fixed t one may easily verify, using known properties of the Fourier-Bessel transform [cf. Watson, Bessel functions, § 14.41], that Φ_1 is $o(r^{-1})$ and hence that 5) is satisfied. One has then the result

$$\Phi(x,y,z,t) = \frac{m(t)}{r(t)} - \frac{m(t)}{r_1(t)} + 2\int_0^\infty dk\,(gk)^{\frac{1}{2}}\int_0^t d\tau\, m(\tau)\sin[(gk)^{\frac{1}{2}}(t-\tau)] e^{k(y+b)} J_0(kR(\tau))$$

$$= \frac{m(t)}{r(t)} - \frac{m(t)}{r_1(t)} +$$

$$+ \frac{1}{\pi}\int_{-\pi}^{\pi} d\vartheta \int_0^\infty dk\,(gk)^{\frac{1}{2}}\int_0^t d\tau\, m(\tau)\sin[(gk)^{\frac{1}{2}}(t-\tau)]\; \cdot \tag{13.49}$$

$$e^{k[y+b(\tau)+i(x-a(\tau))\cos\vartheta + i(z-c(\tau))\sin\vartheta]}.$$

By a more refined analysis of the behavior for large k [cf. Stoker, 1957, pp. 190 - 1] one may establish that Φ is $O(\ )$ and Φ_x and Φ_y are $O(\ )$ as $R \to \infty$.

For some time $t > t_0 \geq 0$, one may write Φ in the form

$$\Phi(x,y,z,t) = 2\int_0^\infty dk\,(gk)^{1/2}\int_0^{t_0} d\tau\, m(\tau)\sin[(gk)^{1/2}(t-\tau)]e^{k[z+b(\tau)]}J_0(kR(\tau))$$

$$+\ \frac{m(t)}{r(t)} - \frac{m(t)}{r_1(t)}$$

$$+\ 2\int_0^\infty dk\,(gk)^{1/2}\int_0^{t-t_0} d\tau\, m(\tau+t_0)\sin[(gk)^{1/2}(t-t_0-\tau)]e^{k[z+b(\tau+t_0)]}J_0(kR(\tau+t_0))$$

$$= \Phi_2(x,y,z,t) + \Phi_3(x,y,z,t).$$

Here the first summand Φ_2 represents the effect at time $t > t_0$ of the action of the source from $t=0$ to $t=t_0$. The remaining terms, Φ_3, are the same as (13.49) with t measured from t_0 $(m(t)=m(t+t_0),\ \ldots)$, and show the effect at time t of the action of the source from $t=t_0$ to $t=t$. (This is, of course, what one would expect from the linearity of the problem and the fact that the choice of $t=0$ is arbitrary.) When $t=t_0$, Φ_3 reduces to

$$\frac{m(t_0)}{r(t_0)} - \frac{m(t_0)}{r_1(t_0)}.$$

Thus

$$\Phi_3(x,0,z,t_0) = 0.$$

This fact provides a basis for Havelock's procedure in similar problems, a procedure originating with Kelvin in the treatment of moving pressure distributions. The idea is roughly as

follows. Divide the path of the source into small segments of time span Δt . If Δt is small enough, the effect of gravity upon the fluid motion produced by the source during this time interval will be negligible, and one may take the boundary condition at the free surface to be $\varphi_t = 0$. The distortion of the surface resulting from the action of the source during this short interval is found and the future behavior of the distortion computed while taking account of gravity. Summing over all Δt and taking the limit leads to the potential function.

The expression (13.49) has been essentially given by Haskind [1946b] and Brard [1948a]. Special choices of $m(t)$ and of the motion of the source lead to cases similar to those treated earlier. Thus, if $m(t) = m \cos \sigma t$ and (a, b, c) is fixed, one has the potential function for a stationary source of oscillating strength, starting to oscillate at $t = 0$. Carrying out the t integration and taking a limit by using, say, the Fourier Integral Theorem (13.1 allow one to derive (13.17). The radiation condition is automatically satisfied. The velocity potential for finite values of t may be written in the form

$$\begin{aligned} \Phi(x,y,z,t) = {} & \frac{m \cos \sigma t}{r} - \frac{m \cos \sigma t}{r_1} \\ & + 2m \cos \sigma t \; PV\!\int_0^\infty \frac{k}{k - \sigma^2/g}\, e^{k(y+b)} J_0(kR)\, dk \\ & - 2m \; PV\!\int_0^\infty \frac{k}{k - \sigma^2/g} \cos (gk)^{\frac{1}{2}} t \; e^{k(y+b)} J_0(kR)\, dk \end{aligned} \qquad (13.50)$$

The leading term in the asymptotic expansion of the last summand gives the last summand of (13.17).

If one takes $m(t) = m$ = a constant, $a(t) = a_0 + u_0 t$, $b(t) = b_0$, $c(t) = c_0$, one obtains the velocity potential for a source suddenly brought into existence at $t = 0$ and moving with constant velocity in the direction Ox [cf. Lunde, 1951, p. 18]. A limit as $t \to \infty$ will give (13.36), the proper boundary conditions at infinite being again automatically satisfied. For finite t the velocity potential in a coordinate system moving with velocity u_0 in direction Ox ($\bar{x} = x - u_0 t$, so that $\bar{\varphi}(x,y,z,t) = \varphi(\bar{x},y,z,t)$) is given by

$$\varphi(\bar{x},y,z,t) = \frac{m}{r} - \frac{m}{r_1} + \frac{m}{\pi}\int_{-\pi}^{\pi} d\vartheta \int_0^{\infty} dk\,(gk)^{\frac{1}{2}} e^{k[y - i\omega(\vartheta)]} \int_0^{t} d\tau \,\sin \tau (gk)^{\frac{1}{2}} e^{iku_0\tau\cos\vartheta}, \tag{13.51}$$

$$\omega(\vartheta) = (\bar{x} - a_0)\cos\vartheta + (z - c_0)\sin\vartheta.$$

The two cases just discussed may be combined by choosing $m(t) = m\cos\sigma t$ and $a(t) = a_0 + u_0 t$, $b(t) = b_0$, $c(t) = c_0$. The modification of (13.51) is simple: a factor $\cos\sigma t$ must be put with the first two terms and a factor $\cos\sigma(t-\tau)$ put at the end of the integral. The asymptotic form as $t \to \infty$ can again be found by use of the Fourier Integral Theorems (13.16) or simple modifications. However, if the resulting formula is written out as principal-value integrals plus other terms, the expression is very unwieldy; it may be found in Havelock [1958]. Use of complex integrals allows one to compress the formula. Let

$$\bar{\Phi}(\bar{x}, y, z, t) = m\cos\sigma t\left(\frac{1}{r} - \frac{1}{r_1}\right) + m\,\mathrm{Re}\, e^{-i\sigma t}\varphi_0, \quad \varphi_0 = \varphi_1 + i\varphi_2.$$

Then

$$\varphi_0 = \frac{2g}{\pi}\int_0^{\gamma} d\vartheta \int_0^{\infty} dk\, F(\vartheta, k) + \frac{2g}{\pi}\int_{\gamma}^{\frac{1}{2}\pi} d\vartheta \int_{L_1} dk\, F(\vartheta, k) + \frac{2g}{\pi}\int_{\frac{1}{2}\pi}^{\pi} d\vartheta \int_{L_2} dk\, F(\vartheta, k),$$

$$F(\vartheta, k) = \frac{k\, e^{k[z + z_0 + i(\bar{x} - a_0)\cos\vartheta]} \cos[k(y - \eta)\sin\vartheta]}{gk - (\sigma + k u_0 \cos\vartheta)^2}, \qquad (13.52)$$

where

$$\tau = u_0\sigma/g$$

$$\gamma = \begin{cases} 0 & \text{if } \tau < \frac{1}{4} \\ \arccos\frac{1}{4\tau} & \text{if } \tau \geq \frac{1}{4} \end{cases}$$

$$\sqrt{gk_1}, \sqrt{gk_3} = \frac{1 - \sqrt{1 - 4\tau\cos\vartheta}}{2\tau\cos\vartheta}\,\sigma,$$

$$\sqrt{gk_2}, -\sqrt{gk_4} = \frac{1 + \sqrt{1 - 4\tau\cos\vartheta}}{2\tau\cos\vartheta}\,\sigma.$$

This potential has been derived by Haskind [1946a], Brard [1948a,b], Hanaoka [1953], Sretenskii [1954], the last with an unfortunate mistake in sign in one term, Eggers [1957], and Havelock [1958]. The latter expresses it in principal-value integrals. Hanaoka, Brard, Eggers, and Sretenskii have each considered the asymptotic form of the surface for large κ. Figure 3 shows qualitatively [cf. Becker, 1958] the curves of equal phase, say the crests, for the various systems of waves formed. The patterns must be completed by reflection in the x-axis.

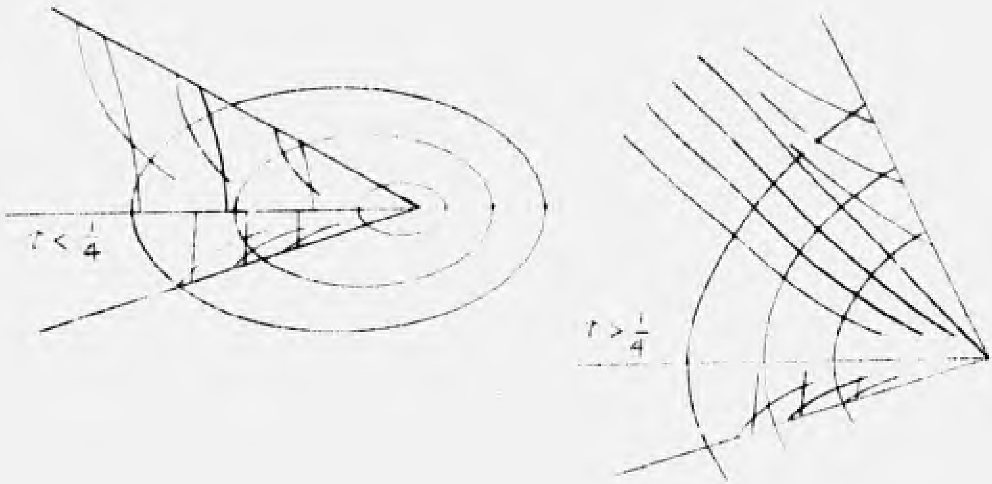

FIGURE 3

Motion of a source on a circular path of radius D may be treated by taking $a = D\cos\omega t$, $c = D\sin\omega t$ in (13.49). For constant ω this problem has been considered by Sretenskii [1946a,b; 1957], Havelock [1950], and Stoker [1957].

One may derive a formula analogous to (13.49) when the source moves in the presence of both a horizontal bottom at $y = -h$ and a free surface. The derivation may be carried out along lines similar to those used in deriving (13.49). The resulting velocity potential is [cf. Lunde, 1951, p. 32]

$$\Phi(x,y,z,t) = \frac{m(t)}{r(t)} + \frac{m(t)}{r_2(t)}$$
$$- 2m(t)\int_0^\infty e^{-kh}\frac{\cosh k(h+b(t))}{\cosh kh}\cosh k(y+h)\, J_0(kR(t))\,dk$$
$$+ 2\int_0^\infty dk\,\sqrt{gk}\,\frac{\cosh k(y+h)}{\cosh^2 kh\sqrt{\tanh kh}} \times \tag{13.53}$$
$$\int_0^t d\tau\,\sin\left[(t-\tau)\sqrt{gk\tanh kh}\right] m(\tau)\cosh k(h+b(\tau))\, J_0(kR(\tau)),$$

where

$$r_2^2 = (x-a(t))^2 + (y+2h+b(t))^2 + (z-c(t))^2.$$

Two-dimensional formulas corresponding to (13.49) and (13.53) may also be derived. They are as follows, with the source and vortex separated for finite depth:

infinite depth: (13.54)

$$f(z,t) = \frac{\Gamma(t)+iQ(t)}{2\pi i}\log(z-c(t)) + \frac{\Gamma(t)-iQ(t)}{2\pi i}\log(z-\bar{c}(t))$$
$$+ \frac{g}{\pi i}\int_0^t \left[\Gamma(\tau)-iQ(\tau)\right]d\tau\int_0^\infty \frac{1}{\sqrt{gk}}\, e^{-ik(z-\bar{c}(\tau))}\sin\left[\sqrt{gk}\,(t-\tau)\right]dk;$$

depth h, source: (13.55)

$$f(z,t) = \frac{Q(t)}{2\pi}\log(z-c(t)) + \frac{Q(t)}{2\pi}\log(z-\bar{c}(t)+2ih)$$
$$+ \frac{Q(t)}{\pi}\int_0^\infty \frac{e^{-kh}}{k\cosh kh}\cosh k(b(t)+h)\cos k(z-a(t)+ih)\,dk$$
$$- \frac{g}{\pi}\int_0^\infty \frac{\operatorname{sech}^2 kh}{\sqrt{gk\tanh kh}}\,dk\int_0^t Q(\tau)\cosh k(b(\tau)+h)\cos k(z-a(\tau)+ih)\cdot$$
$$\sin[\sqrt{gk\tanh kh}\,(t-\tau)]\,d\tau;$$

depth h, vortex: (13.56)

$$f(z,t) = \frac{\Gamma(t)}{2\pi i}\log(z-c(t)) - \frac{\Gamma(t)}{2\pi i}\log(z-\bar{c}(t)+2ih)$$
$$+ \frac{\Gamma(t)}{\pi}\int_0^\infty \frac{e^{-kh}}{k\cosh kh}\sinh k(b(t)+h)\sin k(z-a(t)+ih)$$
$$- \frac{g}{\pi}\int_0^\infty \frac{\operatorname{sech}^2 kh}{\sqrt{gk\tanh kh}}\,dk\int_0^t \Gamma(\tau)\sinh k(b(\tau)+h)\sin k(z-a(\tau)+ih)\cdot$$
$$\sin[\sqrt{gk\tanh kh}\,(t-\tau)]\,dk.$$

Higher-order singularities may be generated by taking derivatives with respect to c . One may transfer to moving coordinates, etc., just as in the three-dimensional case (see Havelock [1949] for (13.54) in moving coordinates). The velocity potential for a steadily moving source of pulsing strength in two dimensions has been given by Haskind [1954, p. 23 ff.], who also gives the asymptotic expressions for large values of $|x|$. When $\tau < 1/4$, there exist one wave far ahead of the moving source propagating in the same direction and three far behind, one propagating in the same direction and two in the opposite direction; when $\tau > 1/4$, there exist two waves far behind propagating in the opposite direction. The analysis for finite depth has been given by Becker [1956].

14. Some simple physical solutions.

In this section we consider periodic waves in an ocean of infinite horizontal extent, either infinitely deep or with a horizontal bottom, in canals, and at an interface. The linearizing parameter ε of section 10α may be taken to be the ratio of amplitude to wave length.

14α. Standing waves in an infinite ocean.

It is appropriate to the physical problem to require that the motion remain bounded everywhere.

Consider first two-dimensional motion. Then, from 13β, the only solutions of the form $\Phi = \varphi\cos(\sigma t + \tau)$ are given by

$$\Phi(x,y,t) = a e^{\nu y}\cos(\nu x+\alpha)\cos(\sigma t+\tau), \quad \nu = \sigma^2/g \tag{14.1}$$

for infinite depth, and

$$\Phi(x,y,t) = a\cosh m_0(y+h)\cos(m_0 x+\alpha)\cos(\sigma t+\tau), \quad m_0\tanh m_0 h - \nu = 0, \tag{14.2}$$

for finite depth.

The corresponding forms of the free surface are given by

$$\eta(x,t) = A\cos(\nu x+\alpha)\sin(\sigma t+\tau)$$

and

$$\eta(x,t) = A\cos(m_0 x+\alpha)\sin(\sigma t+\tau),$$

respectively. These represent standing waves according to our definition in section 7. We recall that $m_0 > \nu$.

It is of interest to examine the streamlines and the paths of the individual fluid particles. The streamlines of the motion can be easily found from

$$\frac{dy}{dx} = \frac{\Phi_y}{\Phi_x} = \cot(\nu x + \alpha)$$

and

$$\frac{dy}{dx} = \frac{\Phi_y}{\Phi_x} = -\tanh m_0(y+h)\cot(m_0 x + \alpha),$$

respectively. The streamlines are then

$$e^{\nu(y-y_m)}\,|\sin(\nu x + \alpha)| = 1$$

$$\tag{14.3}$$

and

$$\sinh m_0(y+h)\,|\sin(m_0 x + \alpha)| = \sinh m_0(y_m + h), \quad 0 \geq y_m \geq h,$$

for infinite and finite depth respectively; here y_m is the lowest point of the streamline. If the fluid is infinitely deep, the streamlines are all congruent. Figure 4 shows three of them for a quarte: wave length and $\alpha = 0$, $\nu = 1$. The vertical line $x = 0$ is also a streamline. If the fluid is of finite depth, the streamlines vary with depth. Figure 5 shows streamlines corresponding to $y_m = 0, -0.5, -0.9$ for $\alpha = 0$, $h = 1$, $m_0 = 1$.

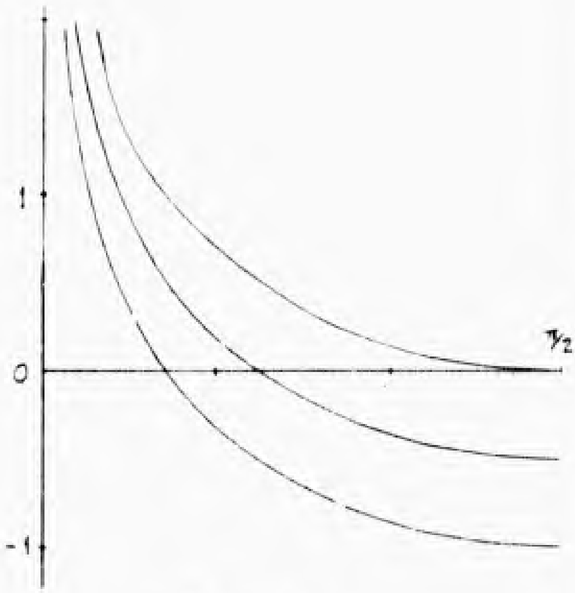

Figure 4

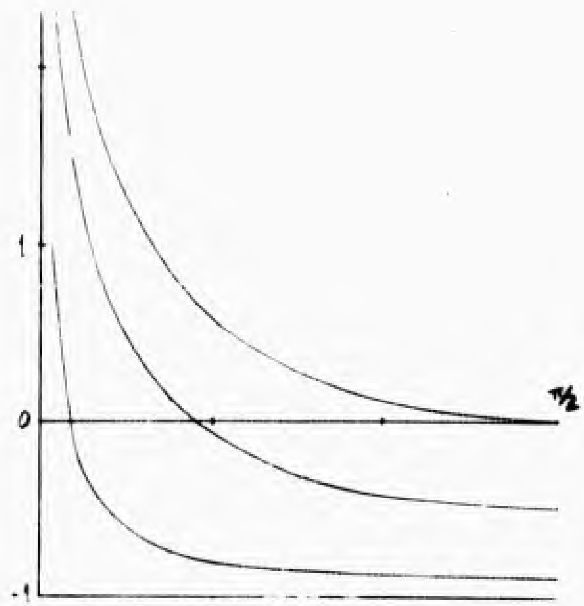

Figure 5

The horizontal line $y = -1$ and the vertical line $x = 0$ are also streamlines.

Since the streamlines are time-independent, they also contain the curves for the trajectories of individual particles. However, the trajectory of an individual particle will be an oscillating motion of small amplitude along a segment of the streamline passing through the point. Thus, in Figure 5 the particles on the bottom simply oscillate back and forth about an equilibrium position, those directly beneath a crest, i.e. at $x = 0$, oscillate vertically, etc. In view of the infinitesimal-wave approximation used in this chapter the streamlines have physical significance for only a small distance above the equilibrium free surface, $y = 0$.

In order to investigate, at least approximately, the behavior of the trajectories more fully, we may replace the actual trajectory by its tangent at an average position, say (x_0, y_0), an approximation consistent with the assumptions made in linearizing. Then the equations describing the trajectory become (setting $\alpha = \tau = 0$)

$$\frac{dx}{dt} = -a\nu e^{\nu y_0} \sin \nu x_0 \cos \sigma t, \quad \frac{dy}{dt} = a\nu e^{\nu y_0} \cos \nu x_0 \cos \sigma t$$

for infinite depth, and

$$\frac{dx}{dt} = -a m_0 \cosh m_0(y_0+h) \sin m_0 x_0 \cos \sigma t, \quad \frac{dy}{dt} = a m_0 \sinh m_0(y_0+h) \cos m_0 x \cos \sigma t$$

for finite depth. The approximate trajectories are then

$$x = x_0 - a\sigma^{-1}\nu e^{\nu y_0} \sin \nu x_0 \sin \sigma t, \quad y = y_0 + a\sigma^{-1}\nu e^{\nu y_0} \cos \nu x_0 \sin \sigma t \tag{14.4}$$

for infinite depth, and

$$x = x_0 - a\sigma^{-1} m_0 \cosh m_0(y_0+h) \sin m_0 x_0 \sin \sigma t, \tag{14.5}$$
$$y = y_0 + a\sigma^{-1} m_0 \sinh m_0(y_0+h) \cos m_0 x_0 \sin \sigma t$$

for finite depth. For infinite depth, the amplitude of oscillation drops off very rapidly as depth of the equilibrium position increases, the ratio of the amplitude at depth y_0 to the amplitude at the free surface being $e^{\nu y_0}$. The same ratio for the case of finite depth is

$$\frac{\sinh^2 m_0(y_0+h) + \sin^2 m_0 x_0}{\sinh^2 m_0 h + \sin^2 m_0 x_0}.$$

Thus, on the bottom, when $y_0 = -h$, the amplitude is zero under the crests and maximum under the nodes. As is evident from the equations of the trajectories, the path lines of particles on the free surface are approximately as in Figure 6.

Figure 6

In order to explain the apparently inconsistent behavior at the nodes one must go to a higher approximation than the linearized theory used in this chapter.

Let us now consider three-dimensional solutions. The standing-wave solutions are of the form

$$\phi(x,y,z,t) = e^{\nu y}\chi(x,z)\cos(\sigma t+\tau) \qquad \text{for finite depth,}$$

or

$$\phi(x,y,z,t) = \cosh m_0(y+h)\,\chi(x,z)\cos(\sigma t+\tau) \qquad \text{for finite depth,}$$

where $\chi(x,z)$ is a solution of

$$\Delta_2\chi + \nu^2\chi = 0 \qquad \text{or} \qquad \Delta_2\chi + m_0\chi = 0,$$

respectively, regular everywhere in $y \le 0$.

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Two particular cases are of especial interest. The first corresponds to separation of variables in rectangular coordinates [see (13.5) and (13.6)]. The solutions are

$$\phi(x,y,z,t) = a e^{\nu y} \cos(k_1 x + \alpha) \cos(k_2 z + \gamma) \cos(\sigma t + \tau), \qquad (14.6)$$
$$k_1^2 + k_2^2 = \nu^2 = \sigma^2/g,$$

for infinite depth, and

$$\phi(x,y,z,t) = a \cosh m_0(y+h) \cos(k_1 x + \alpha) \cos(k_2 z + \gamma) \cos(\sigma t + \tau),$$
$$k_1^2 + k_2^2 = m_0^2, \quad m_0 \tanh m_0 h - \nu = 0, \qquad (14.7)$$

for finite depth. The other solutions result from separating variables in polar coordinates [see (13.7) and (13.8)]. They are

$$\phi(R,\alpha,y,t) = a e^{\nu y} J_n(\nu R) \cos(n\alpha + \delta) \cos(\sigma t + \tau), \quad n = 0, 1, \cdots, \qquad (14.8)$$

for infinite depth, and

$$\phi(R,\alpha,y,t) = a \cosh m_0(y+h) J_n(m_0 R) \cos(n\alpha + \delta) \cos(\sigma t + \tau), \quad n = 0, 1, \cdots, \qquad (14.9)$$

for finite depth. The form of the free surface may be found immediately from $\eta(x,z,t) = -\phi_t(x,o,z,t)/g$. These are all standing waves.

The streamlines and path lines may be found for these two cases with no special difficulty. For the first case for finite depth the streamlines are the intersections of the surfaces

$$|\sin k_1 x|^{k_2^2} = C_1 |\sin k_2 z|^{k_1^2},$$
$$|\sin k_1 x \sin k_2 z| \sinh m_0(y+h) = C_2. \qquad (14.10)$$

The vertical lines, $x = p\pi/k_1$, $z = q\pi/k_2$, passing through the maxima and minima are streamlines. The points on the vertical lines $x = (p+\frac{1}{2})\pi/k_1$, $z = (q+\frac{1}{2})\pi/k_2$ passing through the sattlepoints are all stagnation points. The projection on $y = 0$ of the field of streamlines is indicated qualitatively by Figure 7.

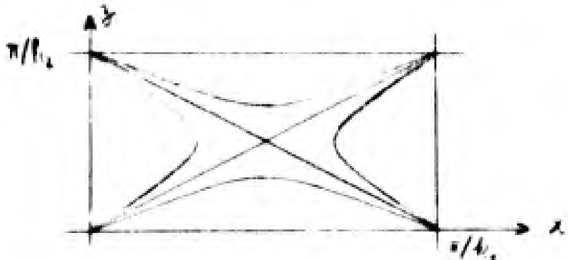


Figure 7

The behavior in a projection on a vertical plane is similar to that for two-dimensional motion.

In the second case above one may easily visualize the streamlines for the case of pure ring waves, n = 0. For finite depth they are given in a plane α = const. by

$$m_0 R\, J_1(m_0 R) \sinh m_0(y+h) = C_1 \tag{14.11}$$

together with the vertical lines at the zeros of $J_1(m_0 R)$. The behavior of the curves is qualitatively similar to that of the two-dimensional case.

In both cases approximations to the path lines can be found as in the two-dimensional case.

14 β. Progressive waves in an infinite ocean.

By taking the proper linear combinations of the standing-wave solutions one may obtain progressive waves. Thus, adding

$$\Phi_1 = a\, e^{\nu y} \cos \nu x \cos \sigma t \quad \text{and} \quad \Phi_2 = a\, e^{\nu y} \sin \nu x \sin \sigma t,$$

one obtains

$$\Phi = a e^{\nu y} \cos(\nu x - \sigma t) \qquad (14.12)$$

which represents a progressive wave moving to the right with velocity

$$c = \frac{\sigma}{\nu} = \frac{g}{\sigma} = \sqrt{\frac{g}{\nu}} = \sqrt{\frac{g\lambda}{2\pi}}\ , \qquad (14.13)$$

where $\lambda = 2\pi/\nu$ is the wave length. Subtracting yields a progressive wave moving to the left. If one takes the coefficient in Φ_1 as a_1 and in Φ_2 as a_2, the sum may be written

$$\Phi = \frac{1}{2} e^{\nu y}\left[(a_1 + a_2)\cos(\nu x - \sigma t) + (a_1 - a_2)\cos(\nu x + \sigma t)\right].$$

This is a superposition of two progressive waves of different amplitudes, one moving to the left and one to the right. If $a_1 = a_2$, a pure progressive wave is obtained, if $a_2 = 0$, one obtains again a standing wave, as a superposition of two progressive waves moving in opposite directions.

For water of finite depth h the corresponding expressions for Φ may be obtained by replacing $e^{\nu y}$ by $\cosh m_0(y + h)$ and ν by m_0. The phase velocity is given by

$$c = \frac{\sigma}{m_0} = \sqrt{\frac{g \tanh m_0 h}{m_0}} = \sqrt{\frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda}}\ . \qquad (14.14)$$

As $h \to \infty$, the velocity approaches that obtained above for deep water. In fact, if $h/\lambda > 0.2$, the velocity is already within 0.1 of the value for deep water. c is an increasing function of λ, but cannot increase indefinitely as in the case of infinitely deep water, for (14.14) implies

$$c < \sqrt{gh} . \tag{14.15}$$

The streamlines for the progressive wave moving to the right are given by

$$e^{\nu y}|\sin(\nu x - \sigma t)| = C \quad \text{and} \quad \sinh m_0(y+h)|\sin(m_0 x - \sigma t)| = C \tag{14.16}$$

for infinite and finite depth, respectively. At a given instant t these have the same shape relative to a crest as the streamlines for a standing wave. However, since they are time-dependent, the path lines for particles do not lie on the streamlines. The path lines may be found approximately for a particle with equilibrium position (x_0, y_0) from the equations

$$\frac{dx}{dt} = \Phi_x(x_0, y_0, t) , \qquad \frac{dy}{dt} = \Phi_y(x_0, y_0, t) .$$

This approximation is consistent with the assumptions made in linearizing the boundary condition, as can be seen by assuming a solution in the form

$$x(t) = x_0 + \varepsilon x_1(t) + \cdots , \quad y(t) = y_0 + \varepsilon y_1(t) + \cdots ,$$

where $\varepsilon = a\sigma\nu/2\pi g$ for infinite depth and $\varepsilon = a\sigma m_0/2\pi g$ for finite depth, substituting in the exact path equations, and retaining only first-order terms.

For infinite depth the particle trajectories are given by

$$x = x_0 - a\nu\sigma^{-1}e^{\nu y_0}\cos(\nu x_0 - \sigma t), y = y_0 - a\nu\sigma^{-1}e^{\nu y_0}\sin(\nu x_0 - \sigma t). \tag{14.17}$$

The particles follow circular orbits of radius $a\nu\sigma^{-1}e^{\nu y_0}$ about the equilibrium position (x_0, y_0); at the top of the orbit they are

moving in the same direction as the wave. The orbital velocity is $a\nu e^{\nu y_0}$, so that the motion dies out quickly as $|y_0|$ increases; for example, at a depth of one wave length the velocity and orbit radius are only 1/535 the value at the free surface. Although the particles at the crest of a wave are moving in the same direction as the wave, their velocity is not necessarily the same and is, in fact, much smaller in view of the assumed smallness of $\varepsilon = (a\nu/c)(\nu/2\pi)$.

For finite depth the orbits are elliptical with the major axis horizontal:

$$\begin{aligned} x &= x_0 - a m_0 \sigma^{-1} \cosh m_0 (y_0 + h) \cos(m_0 x - \sigma t), \\ y &= y_0 - a m_0 \sigma^{-1} \sinh m_0 (y_0 + h) \sin(m_0 x - \sigma t). \end{aligned} \tag{14.18}$$

The particles again trace the orbit in a clockwise direction except that on the bottom they simply oscillate along a horizontal segment. Figure 8 from Ruellan and Wallet [1950] shows the path lines for a variety of cases of superposed waves. The topmost picture shows the orbits for a pure progressive wave moving to the right. The bottom picture is a superposition of progressive waves of equal amplitudes moving in opposite directions, i.e. a pure standing wave. The intermediate cases show superpositions with varying ratios of the amplitudes. The intermediate cases are instructive in that not only path lines, but also streamlines are visible.

Since the progressive-wave solutions are steady with respect to a coordinate system moving with the wave, it clear that we could have obtained a steady-state solution as small motion superposed upon a uniform flow. If we take a complex velocity potential in the form

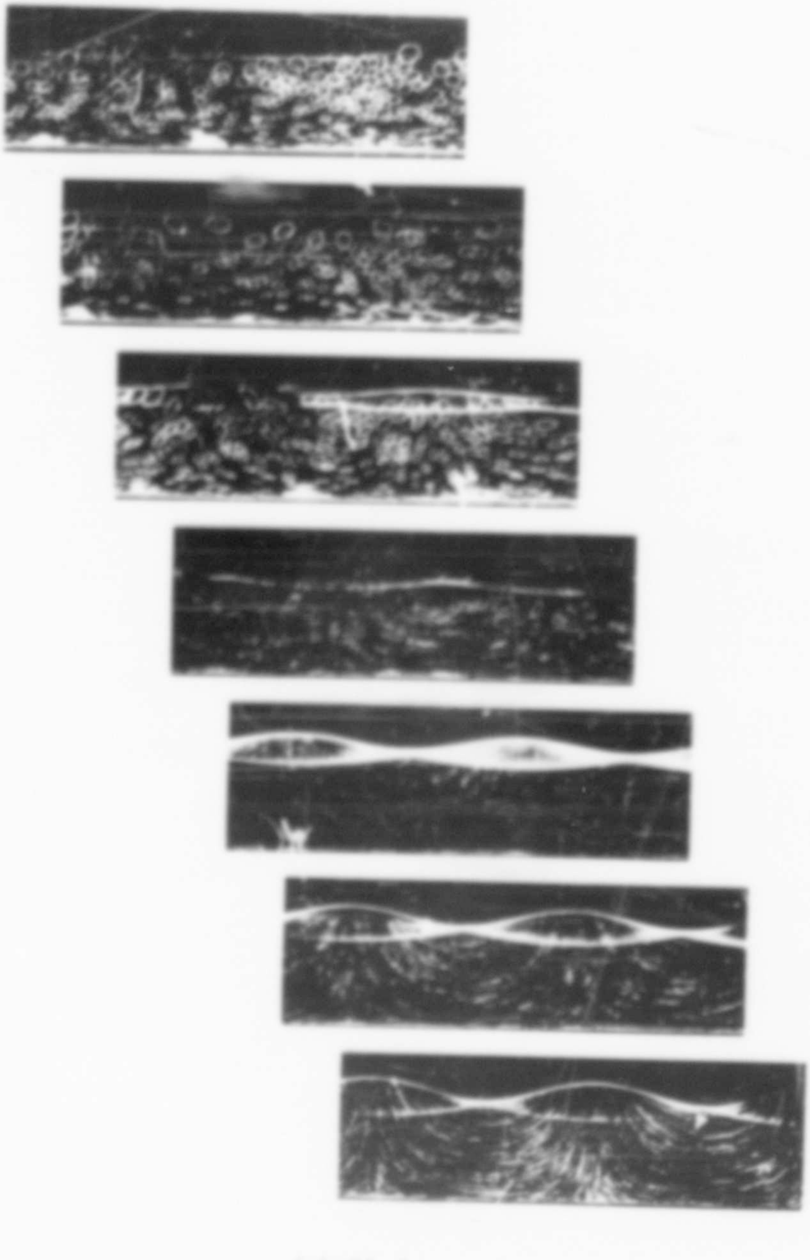


FIGURE 8

$$F(z) = -cz + f(z) \tag{14.19}$$

Then (see equation 11.6) f must satisfy

$$\operatorname{Re}\{igf + c^2 f'\} = 0 \qquad \text{for } y = 0$$

and either $|f'| \to 0$ as $y \to -\infty$ or $\operatorname{Im} f' = 0$ for $y = -h$. The solution for the first case, infinite depth, is given by

$$f = ae^{-i\nu z} = ae^{\nu y}[\cos \nu x - i \sin \nu x] \quad \text{for } \nu = g/c^2 \tag{14.20}$$

The solution for the finite-depth case is given by

$$\begin{aligned} f &= a \cos m_0(z + ih) \\ &= a[\cos m_0 x \cosh m_0(y+h) - i \sin m_0 x \sinh m_0(y+h)], \end{aligned} \tag{14.21}$$

where m_0 must satisfy

$$c^2 m_0 - g \tanh m_0 h = 0 .$$

The same relation is found in (14.14). A real solution does not exist if $c^2/gh > 1$ and in this case there is no wave-like motion consistent with the linearized theory. The streamlines, identical here with the path lines, are obtained from

$$-cy + \psi(x, y) = 0$$

One may replace this equation, consistently with the linearization assumptions [cf. (10.18)]:

$$-cy + \psi(x, y_0) = 0,$$

where y_0 is the mean height of the streamline. Thus, for finite depth, they are given by

$$y = -\frac{a}{c} \sinh m_0 (y_0 + h) \sin m_0 x, \qquad (14.22)$$

an easily constructed family of curves. In the foregoing we have tacitly taken a to be real. However, it may be complex and thus include waves of different phase.

We note finally that (14.8) or (14.9) allow one to construct waves progressing like the spokes of a wheel. However, outwardly progressing waves can be constructed only when the solution involving Y_n is used, and this has a singularity at the origin.

14 γ . Periodic waves in rectangular canals.

Let us suppose that the fluid is contained between the planes $z = 0$ and $z = d$. Then the velocity potential must satisfy the additional conditions

$$\Phi_z(x,y,0,t) = \Phi_z(x,y,d,t) = 0 \qquad (14.23)$$

This condition is automatically satisfied by the two-dimensional waves discussed in 14 α, so that they present no special interest here. However, condition (14.4) does restrict the three-dimensional solutions (14.6) and (14.7), for k_2 must now satisfy (taking $\gamma = 0$).

$$k_2 = \frac{n\pi}{d}, \qquad n = 1, 2, \cdots .$$

Since $k_1^2 + k_2^2 = \nu^2$ or m_0^2, there can be no solution periodic in x unless

$$n < \frac{\nu d}{\pi} \qquad \text{or} \qquad n < \frac{m_0 d}{\pi}, \qquad (14.24)$$

respectively. Hence, for frequencies below a certain critical

frequency σ_1, where

$$\sigma_1 = \sqrt{\frac{\pi g}{d}} \quad \text{or} \quad \sigma_1 = \sqrt{\frac{\pi g}{d} \tanh \frac{\pi h}{d}} \tag{14.25}$$

for infinite or finite depth respectively, there can exist no three-dimensional standing waves in a canal.

Let us form a three-dimensional progressive wave in a canal of finite depth by adding standing-wave solutions:

$$\Phi(x, y, z, t) = a \cosh m_0 (y+h) \cos k_2 z \cos(k_1 x - \sigma t), \quad k_2 = n\pi/d .$$

The velocity of the progressive wave is given by

$$c^2 = \frac{\sigma^2}{k_1^2} = gh\left(1 - \frac{n^2\pi^2}{m_0^2 d^2}\right)^{-1} \frac{\tanh m_0 h}{m_0 h} < gh\left(1 - \frac{n^2\pi^2}{m_0^2 d^2}\right)^{-1} . \tag{14.26}$$

As in the case treated above, there can exist no three-dimensional progressive waves unless $\sigma > \sigma_1$. However, if they exist, their velocity is higher than the velocity of two-dimensional waves of the same frequency.

One may define similarly a sequence of critical frequencies $\sigma_1, \sigma_2, \ldots,$ where

$$\sigma_k = \sqrt{\frac{k\pi g}{d}} \quad \text{or} \quad \sigma_k = \sqrt{\frac{k\pi g}{d} \tanh \frac{k\pi h}{d}} \; ;$$

when $\sigma_k < \sigma < \sigma_{k+1}$, k types of three-dimensional waves are possible with $n = 1, 2, \ldots, k$.

14 δ. Waves at an interface.

Let us now suppose that two fluids are present, one lying over the other. Variables referring to the upper and lower fluids have subscripts 2 and 1 respectively. From (10.7) and (10.8) the linearized boundary conditions for a small disturbance are

$$\Phi_{1y} = \Phi_{2y},$$
$$\rho_1[\Phi_{1tt} + g\Phi_{1y}] = \rho_2[\Phi_{2tt} + g\Phi_{2y}], \tag{14.27}$$

both equations to be satisfied at the equilibrium position of the interface. We shall consider several typical problems.

Let the upper fluid fill the region $y > 0$, and the lower fluid the region $y < 0$. We require of a solution that

$$|\text{grad}\ \Phi_1| \to 0 \text{ as } y \to -\infty \quad \text{and} \quad |\text{grad}\ \Phi_2| \to 0 \text{ as } y \to +\infty.$$

In looking for a standing-wave solution, one may, following 14α, take

$$\Phi_1 = a_1 e^{my}\varphi(x,z)\cos(\sigma t + \tau), \quad \Phi_2 = a_2 e^{-my}\varphi(x,z)\cos(\sigma t + \tau),$$

where the relation between a_1 and a_2 and m and σ is to be determined by (14.27), and φ satisfies

$$\Delta_2\varphi + m^2\varphi = 0.$$

The first equation (14.27) gives immediately that

$$a_1 + a_2 = 0.$$

The second one gives the relation

$$\sigma^2 = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} mg. \tag{14.28}$$

The equation of the interface may be obtained from (10.8):

$$\eta(x,z,t) = \frac{a_1 m}{\sigma} \varphi(x,z) \sin(\sigma t + \tau).$$

Since $a_1 = -a_2$, there is a discontinuity in u (and w if the motion is three-dimensional) as one crosses the interface.

The special choices of $\varphi(x,z)$ made in 14β may, of course, also be made here. In particular, one may make progressive and standing waves. If one forms two-dimensional progressive waves at the interface, one finds for the velocity

$$c^2 = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \frac{g}{m}. \qquad (14.29)$$

If one assumes the fluids bounded above and below by planes $y = h_2$ and $y = -h_1$, respectively, a similar calculation shows

$$\sigma^2 = \frac{\rho_1 - \rho_2}{\rho_1 \coth mh_1 + \rho_2 \coth mh_2} gm \qquad (14.30)$$

It is clear from (14.28) and (14.30) that these solutions exist only if $\rho_2 < \rho_1$. The case $\rho_2 > \rho_1$ will be discussed later.

A more complicated problem of this type is the following (cf. Lamb [1932, § 231], Greenhill [1887]). Suppose there is a solid horizontal bottom at $y = -h$, an interface at $y = -d$ and a free surface at $y = 0$. Then, in addition to (14.27) at $y = -d$, Φ_1 and Φ_2 must satisfy

$$\Phi_{2tt} + g\,\Phi_{2y} = 0 \quad at \quad y = 0, \qquad \Phi_{1y} = 0 \quad at \quad y = -h.$$

If one seeks solutions of the form

$$\Phi_2 = (a_2 \cosh my + b_2 \sinh my)\,\varphi(x,z)\cos(\sigma t+\tau),$$

$$\Phi_1 = a_1 \cosh m(y+h)\,\varphi(x,z)\cos(\sigma t+\tau),$$

substitution in the various boundary conditions yields the following relation between σ and m:

$$\left(\frac{\sigma^2}{gm}\right)^2\left[\rho_1 \coth md \coth m(h-d) + \rho_2\right]$$

$$- \frac{\sigma^2}{gm}\rho_1\left[\coth md + \coth m(h-d)\right] + (\rho_1-\rho_2) = 0. \qquad (14.31)$$

If $\rho_2 < \rho_1$, one may establish that there exist two positive solutions for σ^2 for a given m, so that two possible frequencies are possible for a given wave pattern. If the bottom fluid is taken infinitely deep, one replaces $\coth m(h-d)$ by 1 in (14.31) and the two solutions simplify to

$$\sigma_1^2 = gm, \qquad \sigma_2^2 = gm\frac{\rho_1-\rho_2}{\rho_1 \coth md + \rho_2} < \sigma_1^2. \qquad (14.32)$$

The first solution, σ_1, is the same as would be obtained if the two fluids were identical (and there is no discontinuity in u and w at the interface); the second, σ_2, is interpreted below. The inequality $\sigma_2^2 < \sigma_1^2$ holds in general, and one may establish

$$\frac{\sigma_2^2}{gm} < \{\tanh md, \tanh m(h-d)\} \le \frac{\sigma_1^2}{gm} \le \min\left\{1, \frac{\rho_1}{\rho_2}\tanh mh\right\}. \qquad (14.33)$$

If one computes the ratio of the amplitude of the disturbance at the interface to that at the free surface, one finds, no matter whether h is finite or not,

$$\cosh md - \frac{gm}{\sigma^2}\sinh md. \qquad (14.34)$$

An examination of the roots of (14.31) shows that the ratio (14.34)

is negative for the smaller of the two roots and positive for the larger. Thus, in the solution associated with the smaller roots, a maximum of the disturbance at the interface is associated with a minimum of that at the free surface, and vice versa. On the other hand, with the larger root the maxima and minima go together. For the values given in (14.32), the ratio becomes

$$e^{-md} \quad \text{and} \quad -\frac{\rho_1}{\rho_1-\rho_2}\, e^{md}, \tag{14.35}$$

respectively. We note that, although the first ratio is < 1, the second is in absolute value > 1 if $\rho_2(1+e^{md}) > \rho_1 > \rho_2$, a condition satisfied if ρ_1 is only slightly greater than ρ_2. In fact, the ratio may become very large.

For a given wave length and amplitude of the wave at the free surface one may also compare the amplitudes of the two different modes of motion at the interface. If A_i is the amplitude associated with the frequency σ_i, then for the case $h = \infty$ one finds

$$\left|\frac{A_2}{A_1}\right| = \frac{\rho_2}{\rho_1-\rho_2}\,\frac{1+\tanh md}{1-\tanh md},$$

which may be either less than or greater than 1.

It is of some interest to examine somewhat further the solution associated with the smaller root σ_2 of (14.31). Then, since $a_2/b_2 = gm/\sigma^2$, the inequality (14.33) implies that there exists an h_0 with $0 < h_0 < d$ such that

$$\frac{\sigma_2^2}{gm} = \frac{b_2}{a_2} = \tanh mh_0 < \tanh md < 1.$$

and that

$$\Phi_2 = \sqrt{a_2^2 - b_2^2}\,\cosh m(y+h_0)\,\varphi(x,z)\cos(\sigma t+\tau).$$

Thus the part of the top fluid between $y = 0$ and $y = -h_0$ behaves as if there were a solid boundary at $y = -h_0$; and, of course, the fluid between $y = -h_0$ and $y = -h$ as if it were between solid boundaries. If one has selected solutions for φ which can be combined to form a progressive plane wave, then one may conclude that the velocity $c_2 = \sigma_2/m$ associated with this mode of motion has an upper bound:

$$c_2 = \sqrt{\frac{g}{m}\tanh mh_0} < \sqrt{gd}.$$

In fact, when $h = \infty$, one may verify immediately from (14.32) that

$$c_2 = \sqrt{\frac{g}{m}\tanh md \; \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2 \tanh md}} \leq \sqrt{g d \, \frac{\rho_1 - \rho_2}{\rho_1}} = c_{2\max}.$$

Thus for $h = \infty$ a progressive wave travelling faster than $c_{2\max}$ will consist of only the one mode of motion, i.e. the one associated with σ_1. If $c < c_{2\max}$, there may be two modes of motion excited. This fact is associated with the phenomenon of "dead-water" resistance of ships (see Lamb [1916a], Ekman [1904], Sretenskii [1934]).

For superposed fluids one may also find solutions analogous to (14.20) and (14.21). Let us suppose that the first (upper) fluid flows to the left with mean velocity c_2 and the second with mean velocity c_1. We wish to find the possible steady periodic profiles of the interface, assuming as usual that the disturbance is small. The complex velocity potential for each fluid is taken in the form

$$F_1(z) = -c_1 z + f_1(z), \quad F_2(z) = -c_2 z + f_2(z). \tag{14.30}$$

The conditions to be satisfied at the mean common boundary, $y = 0$, are:

$$c_1^{-1}\,\mathrm{Im}\, f_1 = c_2^{-1}\,\mathrm{Im}\, f_2\,, \tag{14.37}$$

$$\rho_1 c_1^{-1}\,\mathrm{Re}\{ig f_1 + c_1^2 f_1'\} = \rho_2 c_2^{-1}\,\mathrm{Re}\{ig f_2 + c_2^2 f_2'\}.$$

If each fluid extends infinitely far vertically, then

$$f_1 = a_1 e^{-imz}\,, \qquad f_2 = a_2 e^{imz}$$

give a steady-state solution if

$$a_1/c_1 = -\bar{a}_2/c_2$$

and

$$m = \frac{g(\rho_1 - \rho_2)}{\rho_1 c_1^2 + \rho_2 c_2^2} > 0, \tag{14.38}$$

where $\bar{a}_2$ is the complex conjugate of a_2. If the upper fluid is bounded by $y = h_2$ and the lower by $y = -h_1$, then the solution is

$$f_1 = a_1 \cos m(z + ih_1)\,, \quad f_2 = a_2 \cos m(z - ih_2),$$

where, letting $a_k = \alpha_k + i\beta_k$, $k = 1, 2$,

$$\frac{\alpha_1}{c_1}\sinh mh_1 = -\frac{\alpha_2}{c_2}\sinh mh_2\,, \quad \frac{\beta_1}{c_1}\cosh mh_1 = \frac{\beta_2}{c_2}\cosh mh_2$$

and (14.39)

$$m = \frac{g(\rho_1 - \rho_2)}{\rho_1 c_1^2 \coth mh_1 + \rho_2 c_2^2 \coth mh_2}.$$

In either case the equation of the interface is given by

$$y = \frac{1}{c_k}\psi_k(x, 0).$$

Sretenskii [1952b] has considered a three-dimensional analogue of the above problem in which the direction of flow of one of the

fluids makes an angle ϑ with that of the other. Thus, take velocity potentials of the following form:

$$\begin{aligned} \Phi_2(x,y,z) &= -c_2(x\cos\vartheta + z\sin\vartheta) + \varphi_2(x,y,z), \\ \Phi_1(x,y,z) &= -c_1 x + \varphi_1(x,y,z). \end{aligned} \tag{14.40}$$

The following are the boundary conditions at the interface $\eta(x,z)$ for small disturbance:

$$\begin{aligned} &\varphi_{2y}(x,0,z) + c_2(\eta_x\cos\vartheta + \eta_z\sin\vartheta) = 0, \quad \varphi_{1y}(x,0,z) + c_1\eta_x = 0 \\ &g(\rho_1-\rho_2)\eta = \rho_1 c_1 \varphi_{1x}(x,0,z) - \rho_2 c_2[\varphi_{2x}(x,0,z)\cos\vartheta + \varphi_{2z}(x,0,z)\sin\vartheta]. \end{aligned} \tag{14.41}$$

For a solution in the form

$$\varphi_1 = A_1 e^{my}\cos(k_1 x + k_2 z), \quad \varphi_2 = A_2 e^{-my}\cos(k_1 x + k_2 z),$$
$$k_1^2 + k_2^2 = m^2,$$

the following relations must hold

$$\frac{A_2}{A_1} = -\frac{c_2}{c_1}\,\frac{k_1\cos\vartheta + k_2\sin\vartheta}{k_1}$$

and (14.42)

$$\rho_1 c_1^2 k_1^2 + \rho_2 c_2^2 (k_1\cos\vartheta + k_2\sin\vartheta)^2 = gm(\rho_1 - \rho_2).$$

These reduce to (14.38) for $\vartheta = 0$, $k_1 = m_1$ as they should. The equation for the interface is

$$y = -A_1\frac{m}{k_1 c_1}\sin(k_1 x + k_2 z) \tag{14.43}$$

Sretenskii studies the properties of the solution in more detail.

As a further extension of the preceding cases one may consider a time-dependent disturbance at the interface between two fluids flowing at different velocities. This will be treated in the section on stability of motion.

A natural generalization of the two-fluid system is the n-fluid system (see Greenhill [1887]) and then the heterogeneous fluid with density given as a series

$$\rho(x,y,z,t) = \rho_0(y) + \varepsilon\rho^{(1)}(x,y,z,t) + \varepsilon^2\rho^{(2)} + \cdots .$$

If one assumes a similar expansion for p and expansions for u, v, w, and η starting with ε, one may derive easily the linearized equations. These, discussion of some periodic solutions, and references to the literature may be found in Lamb [1932, § 235]. Groen [1948] has shown that the period for simple harmonic motion in the linearized problem is a monotonic increasing function of the wave length starting with the minimum $2\pi\sqrt{-\rho_0(y)/g\rho_0'(y)}$ for $\lambda = +0$. Groen [1950] discusses properties of internal waves in an expository way and gives further references to the more recent literature. For some pertinent theorems about waves in heterogeneous fluids see section 32β.

15. Group velocity and the propagation of disturbances and of energy.

In the last section we considered periodic waves at a free surface or interface. In this section we wish to consider waves of a given but fairly general initial form and study the way in which they propagate. Although this will entail writing down the solution to a particular initial-value problem, this is of only incidental interest, the chief interest being in the history of the form of the free surface or interface. Initial-value problems as such will be treated in more detail later on. In fact, the remarks below apply equally well to other initial-value problems, for example, an initial distribution of velocity on the surface. What is essential is the resolution of the subsequent motion into a set of waves moving to the right and of ones moving to the left, as in (15.2).

The property of the fluid and its boundaries which is most important for this investigation is the functional relation between the frequency σ and the wave number k. The earlier parts of this chapter have shown that considerable variation is possible in the form of $\sigma(k)$. The two-fluid example with both free surface and interface gave a doubly valued function. A multiply valued function could have been obtained with more layers. However, each branch, or the branch, is a decreasing function of k, approaching zero as $k \to \infty$. When surface tension is taken into account [see section 24], the form of $\sigma(k)$ for large k changes; it then becomes an increasing function, behaving like $k^{3/2}$. If h is large enough, $\sigma(k)$ decreases initially, i.e. for $k < k_m$, reaches a minimum at k_m and then increases; if h is small enough $\sigma(k)$ is everywhere increasing. It will be convenient to extend the definition of $\sigma(k)$ to negative k by setting $\sigma(-k) = -\sigma(k)$.

15α . The propagation of an initial elevation.

Let us suppose that at time $t = 0$ the free surface is given by $y = \eta(x,0)$ and that the fluid is at rest. How does the free surface behave subsequently? One may conveniently think of this as an initial humping up of the fluid near one point, but this is not essential. We shall also suppose that $\eta(x,0)$ is sufficiently restricted to allow a Fourier-integral representation. In part of what follows we shall also assume it to be square integrable, i.e. the total available energy is finite, and on occasion that $x\eta$ is square integrable. Let

$$\eta(x,0) = \int_0^\infty [C(k)\cos kx + S(k)\sin kx]\,dk$$
$$= \int_{-\infty}^\infty e^{-ikx}E(k)\,dk = \tfrac{1}{2}\,\mathrm{Re}\int_0^\infty e^{-ikx}E(k)\,dk, \qquad (15.1)$$

where

$$C(k) = \frac{1}{\pi}\int_{-\infty}^{\infty} \eta(x,0)\cos kx\, dx, \quad S(k) = \frac{1}{\pi}\int_{-\infty}^{\infty} \eta(x,0)\sin kx\, dx,$$

$$E(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \eta(x,0)e^{ikx}\, dx = \frac{1}{2}\left[C(k) + i\, S(k)\right].$$

We shall call E(k) the <u>spectrum</u> of $\eta(x,0)$. Note that $E(-k) = E^*(k)$, the complex conjugate of E(k) [we change notation temporarily in order to avoid conflict with the notation for averages introduced below].

A formal solution for Φ and $\eta(x,t)$ may be written down immediately:

$$\begin{aligned}
\Phi(x,y,t) &= -\int_0^{\infty} \frac{\sigma(k)}{k} Y(y)\left[C(k)\cos kx + S(k)\sin kx\right]\sin \sigma t\, dk \\
&= -\int_{-\infty}^{\infty} \frac{\sigma(k)}{k} Y(y) E(k) e^{-ikx} \sin \sigma t\, dk \\
&= \frac{1}{2}i\int_{-\infty}^{\infty} \frac{\sigma(k)}{k} Y(y) E(k)\left[e^{-i(kx-\sigma t)} - e^{-i(kx+\sigma t)}\right] dk \qquad (15.2)
\end{aligned}$$

$$\begin{aligned}
\eta(x,t) &= \int_0^{\infty} \left[C(k)\cos kx + S(k)\sin kx\right]\cos \sigma t\, dk \\
&= \int_{-\infty}^{\infty} e^{-ikx} E(k)\cos \sigma t\, dk = \frac{1}{2}\int_{-\infty}^{\infty} E(k)\left[e^{-i(kx-\sigma t)} + e^{-i(kx+\sigma t)}\right] dk.
\end{aligned}$$

Here $Y(y) = \cosh k(y+h)/\sinh kh$ for a single fluid of depth h, $Y(y) = e^{|k|y}\operatorname{sgn} k$ for infinite depth (the peculiar modification of Y for $h = \infty$ is necessary for $k < 0$). However, more general situations are allowable in which, for example, $\eta(x,t)$ describes an interface. The choice of an expression for Φ has been based upon the kinematic boundary condition $\Phi_y(x,0,t) = \eta_t(x,t)$ in order not to exclude the possibility of surface tension. For simplicity we also restrict ourselves to single-valued σ's. For more complicated problems, such as the two-fluid problem with both free surface and

interface discussed in § 148, the freedom to fix both $\eta_1(x, 0)$ and $\eta_2(x, 0)$ independently requires the determination of two spectra for each surface with relations between them set by (14.34). The remarks below will still apply to motion resulting from each spectrum separately. Finally, we note that a statement concerning specific conditions to be satisfied by $\eta(x, 0)$ for the case of a single free surface may be found in a paper by Kampé de Fériet and Kotik [1953].

It is clear from (15.2) that one may express $\eta(x,t)$ as a sum of two functions, one, say $\eta_R(x,t)$, representing a superposition of waves moving to the right, the other, η_L, waves moving to the left. We consider only η_R since similar remarks apply to η_L with x replaced by -x. The spectrum of η_R is given by $\frac{1}{2} E(k) e^{i\sigma(k)t}$ so that clearly $\sigma(k)$ plays an important role in the change of shape of η_R. Since each harmonic component in η_R is moving to the right with velocity $\sigma(k)/k$, and since this is not a constant in the cases we have been considering, the different components will move with different velocities and we shall expect η_R to change its shape with time, even though moving as a whole to the right.

In order to get some idea of the overall motion it is reasonable to try to compute an average position of $\eta_R(x,t)$ and find how this moves. One must first decide how to define the average position. One possibility, which, as we shall see presently, is unsatisfactory is to use η_R itself as the weighting function, i.e. to define

$$\bar{x}_R(t) = \int_{-\infty}^{\infty} x\,\eta_R(x,t)\,dx \Big/ \int_{-\infty}^{\infty} \eta_R(x,t)\,dx$$

when this exists. An easy computation shows that

$$\bar{x}_R(t) = \bar{x}_R(0) + \sigma'(0)t,$$

i.e. the average motion is, on this definition, independent of the form of $\sigma(k)$ except near $k = 0$. For deep-water gravity waves $\sigma'(0) = \infty$; for depth h, $\sigma'(0) = \sqrt{gh}$, the maximum velocity (see equation (14.15)). In conformity with the above one may define the "spread" of the hump to be

$$\int_{-\infty}^{\infty} [x - \bar{x}_R(t)]^2 \eta_R(x,t) dx \Big/ \int_{-\infty}^{\infty} \eta_R(x,t) dx .$$

A computation shows that this remains constant in time, when it exists. This definition of average is unsatisfactory, as could have been expected inasmuch as the weighting function can become negative. We note in passing that

$$\int_{-\infty}^{\infty} \eta_R(x,t)\, dx = \int_{-\infty}^{\infty} \eta_R(x,0)\, dx,$$

an expression of conservation of mass.

Another possible weighting function without this shortcoming, but still allowing ease of computation, is $\eta_R^2(x,t)$. We note first that

$$\int_{-\infty}^{\infty} \eta_R^2(x,t)\, dx = \int_{-\infty}^{\infty} \eta_R^2(x,0) dx = \tfrac{1}{2}\pi \int_{-\infty}^{\infty} E(k) E^*(k)\, dk .$$

Let us define two averages, one for functions of x:

$$\bar{f}(t) = \int_{-\infty}^{\infty} f(x)\, \eta_R^2(x,t) dx \Big/ \int_{-\infty}^{\infty} \eta_R^2(x,t)\, dx,$$

and one for functions of k:

$$\bar{\phi} = \int_{-\infty}^{\infty} \phi(k)\, E(k) E^*(k) dk \Big/ \int_{-\infty}^{\infty} E(k) E^*(k)\, dk .$$

Then, assuming that the various quantities in question exist, one finds, using well known theorems on Fourier transforms [see, e.g., Bochner and Chandrasekharan, 1949, ch. IV, § 2],

$$\overline{x}_R(t) = \overline{x}_R(0) + \overline{\sigma'}\, t \tag{15.3}$$

and

$$\overline{[x - \overline{x}_R(t)]^2} = \overline{[x - \overline{x}_R(0)]^2} + t\left\{ \overline{\sigma' \left[i \log \frac{E^*}{E} \right]'} - \overline{\sigma'}\, \overline{\left[i \log \frac{E^*}{E} \right]'} \right\} + t^2 \left\{ \overline{\sigma'^2} - \overline{\sigma'}^2 \right\}. \tag{15.4}$$

Thus, on this definition the average position of η_R moves to the right with constant velocity $\overline{\sigma'}$ and the hump spreads according to a quadratic law. We note that the coefficient of t^2 is positive except if σ' is a constant, when it vanishes. It may become infinite, and, in fact, does so for infinitely deep water if the gravest modes are present, i.e., if $\int \eta_R\, dx \neq 0$. The coefficient of t vanishes if σ' is constant or if $[i \log E^*/E]'$ is constant; the latter will occur if $\eta(x, 0)$ is either symmetric or antisymmetric about some point x_0, but this does not exhaust all possibilities. The sign of this term does not seem to be determined, so that the spread of the hump may conceivably decrease before starting to increase.

Investigations of the motion of the average position of the hump and of its spread give only a rather crude picture of its behavior. By other methods outlined below one may obtain further insight into the motion.

We begin by applying the analysis of the average motion to that part of η_R resulting from only a narrow band in its spectrum. Let

$$\eta_R(x,t;k_0,\epsilon) = \mathrm{Re}\int_{k_0-\epsilon}^{k_0+\epsilon} \frac{1}{2}E(k)\,e^{-i(kx-\sigma(k)t)}\,dk \tag{15.5}$$

We shall call this a wave packet. The average position satisfies

$$\bar{x}_R(t;k_0,\epsilon) = \bar{x}_R(0;k_0,\epsilon) + \overline{\sigma'}(k_0,\epsilon)\,t,$$

where $\overline{\sigma'}(k_0,\epsilon)$ is now the average of $\sigma'(k)$ over the narrow band $[k_0-\epsilon, k_0+\epsilon]$. The narrower the band, the closer $\overline{\sigma'}(k_0,\epsilon)$ is to $\sigma'(k_0)$, assuming the latter continuous. As a limiting case we shall say that the wave packet resulting from an infinitesimal band about k_0 moves with velocity $\sigma'(k_0)$. It is customary to call $\sigma'(k)$ the group velocity. This is the same as the phase velocity $\sigma(k)/k$ only if $\sigma = ak$. A wave packet will spread with passage of time unless the two velocities are equal, for (15.4) is applicable to the wave packet with the restricted definition of average. As might be expected, the smaller the width of the band, the smaller the coefficient of t^2 and the smaller the rate of growth. However, as we shall see below, the initial spread may be wide for a narrow band.

The wave packet (15.5) may also be investigated by a different method. Let us expand $\sigma(k)$ in the first few terms of a Taylor series about k_0:

$$\sigma(k) = \sigma(k_0) + \sigma'(k_0)(k-k_0) + \frac{1}{2}\int_{k_0}^{k}\sigma''(\kappa)(k-\kappa)\,d\kappa. \tag{15.6}$$

We may then write

$$\begin{aligned}\eta_R(x,t;k_0,\epsilon) &= \mathrm{Re}\,\frac{1}{2}e^{-i[k_0x-\sigma(k_0)t]}\left\{\int_{k_0-\epsilon}^{k_0+\epsilon}E(k)\,e^{-i[x-\sigma'(k_0)t](k-k_0)}\,dk\right.\\ &\quad\left.+\int_{k_0-\epsilon}^{k_0+\epsilon}E(k)\,e^{-i[x-\sigma'(k_0)t](k-k_0)}\left[\exp\left(-\tfrac{1}{2}it\int_{k_0}^{k}\sigma''(\kappa)(k-\kappa)\,d\kappa\right)-1\right]dk\right\} \qquad (15.7)\\ &= \mathrm{Re}\,\frac{1}{2}e^{-i[k_0x-\sigma(k_0)t]}\,M(x-\sigma'(k_0)t;k_0,\epsilon)+R.\end{aligned}$$

Using the inequality $|e^{iu} - 1| \leq |u|$, one finds

$$|R| \leq \frac{1}{4} t \varepsilon^3 \max_{|k-k_0|<\varepsilon} |E(k)| \max_{|k-k_0|<\varepsilon} |\sigma''(k)|. \tag{15.8}$$

The remainder can thus be made small by taking ε or t small enough. However, once ε is fixed, R will eventually become large as t increases. Let us suppose, however, that t and ε are small enough so that the first term determines the main features of the motion. The first factor represents a periodic wave of wave number k_0 moving with its phase velocity $\sigma(k_0)/k_0$. The second factor, determining the amplitude of the first, represents a profile being translated to the right with velocity $\sigma'(k_0)$. Thus one may say that the gross outline of the surface is moving to the right with the group velocity. One may see this more clearly if one assumes ε small enough so that we may take E(k) as constant over the band width. Then

$$M(x - \sigma'(k_0)t; k_0, \varepsilon) = E(k_0) \frac{\sin (x - \sigma'(k_0)t)\varepsilon}{x - \sigma'(k_0)t},$$

and $\eta_R(x,t; k_0, \varepsilon)$ appears approximately as in Figure 9. Here the dotted enveloping curves represent $\pm \frac{1}{2} M$ and move to the right with velocity $\sigma'(k_0)$, whereas the inscribed solid curves represent the first factor and move to the right with phase velocity

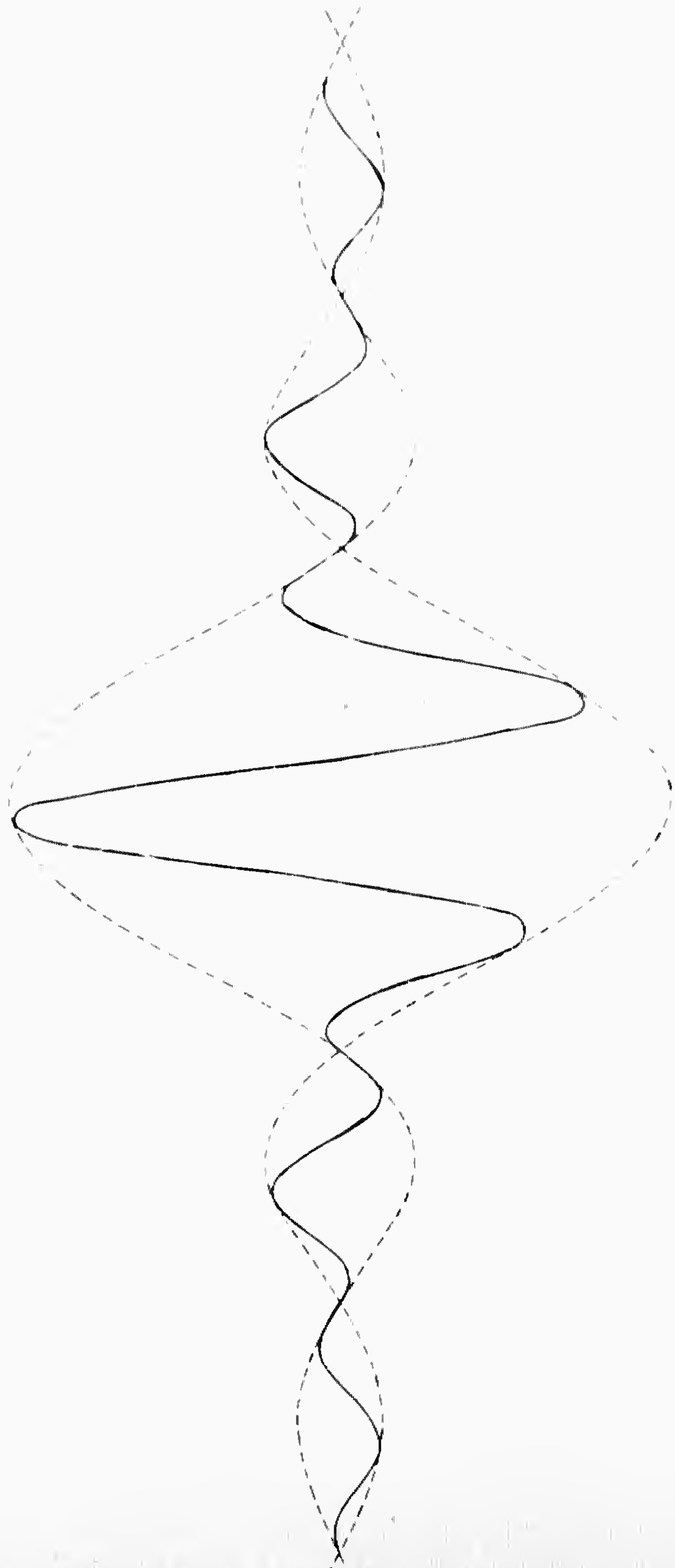


Figure 9

$\sigma(k_0)/k_0$. The whole moves as a fixed pattern only if the two velocities are equal. Otherwise, assuming $\sigma'(k_0) < \sigma(k_0)/k_0$, the inscribed curves will progress through the wave packet, gradually disappearing at the right. For a very narrow band the packet will spread wide before its first zero on either side of the maximum.

A disadvantage of this last analysis is that it becomes less and less accurate as t becomes large. However, there exists another approximation to $\eta_R(x,t)$ for large values of t which helps to complete the picture. This ultimate behavior of η_R can to some extent be predicted from the analysis of the average motion of a wave band. If we think of η_R as made up of the contributions from a number of narrow wave bands, we know that each contribution is moving with the average group velocity of the band. Thus after some time we shall expect that these various contributions will have separated from one another, with the bands about the gravest modes, which travel fastest, having progressed the furthest. This prediction will be confirmed.

What is needed for this final approximation is an asymptotic expansion for large t. It is convenient to express η_R in the slightly altered form

$$\eta_R(x,t) = \frac{1}{2}\int_{-\infty}^{\infty} E(k)\, e^{-i\left[k\frac{x}{t} - \sigma(k)\right]t}\, dk \tag{15.9}$$

and to consider it as depending upon the two parameters x/t and t. Then for each value of x/t we shall give an expansion for large values of t. For a derivation of the expansi . we refer to Stoker [1957, § 6.8] or Erdélyi [1956, § 2.9] .

Let the functions $k_r(x/t)$, $r = 1, 2, \ldots, n$, be defined by

$$\sigma'(k_r) = x/t, \tag{15.10}$$

i.e. we allow the possibility of several roots. In the situation of interest to us there will be either one or two roots, or none. The asymptotic expression for η_R is then given by

$$\eta_R(x,t) = \mathrm{Re} \sum_r \tfrac{1}{2} E(k_r) \left[\frac{2\pi}{t|\sigma''(k_r)|} \right]^{1/2} e^{-i[k_r x - \sigma(k_r)t - \frac{1}{4}\pi \operatorname{sgn} \sigma''(k_r)]}$$

$$+ \mathrm{Re} \sum_r \tfrac{1}{2} E(k_r) \frac{1}{\sqrt{3}} \Gamma(\tfrac{1}{3}) \left[\frac{6}{t|\sigma'''(k_r)|} \right]^{1/3} e^{-i[k_r x - \sigma(k_r)t]} + O(t^{-2/3}), \tag{15.11}$$

where the first summation is over all values of r for which $\sigma''(k_r) \neq 0$ and the second over all k_r for which $\sigma''(k_r) = 0$ but $\sigma'''(k_r) \neq 0$; further terms would be necessary for values of r for which both vanish but this will not occur in our examples. If some $k_r = 0$, then the corresponding term must be multiplied by $\frac{1}{2}$. For a value of x/t for which no solution to (15.10) exists, it is easy to show by a change of variables in (15.9), say $u = kx/t - \sigma(k)$, and integration by parts that $\eta_R(x,t) = O(t^{-1})$.

Let us examine in some detail the implications of one term of (15.11), say $r = 1$, for the motion of η_R; if several terms are present for a given value of x/t one must superpose the resultant motions.

If x/t is held constant while t increases, then clearly one must set $x = \sigma'(k_1)t$, i.e. we are examining η_R from the standpoint of an observer moving with group velocity $\sigma'(k_1)$. Since the coefficient of the harmonic term is $t^{-1/2}$ times a function of k_1, which

is being held constant, the gross outline of η_R will appear constant in form, but decreasing in amplitude because of $t^{-1/2}$ However, just as in the analysis of (15.7), there is a harmonic ov wave number k_1 moving through the gross outline with phase velocity $\sigma(k_1)/k_1$. The amplitude of the gross outline is proportional to $E(k_1)$, but also depends now upon $\sigma''(k_1)$, in contrast to the situation for small t according to (15.7)

If the value of x/t is such that $\sigma''(k_1) = 0$, then one must examine a term from the second summation in (15.11) It is evident that the interpretation is the same except that σ''' occurs in place of σ'' and that the amplitude decreases more slowly because of the $t^{-1/3}$. This situation can happen, for example, in the case of gravity waves in water of depth h for $x = t\sqrt{gh}$. Then $k_1(\sqrt{gh}) = 0$, $\sigma''(0) = 0$, and $\sigma'''(0) = -h^2\sqrt{gh}$ This also occurs for combined gravity-capillary waves when the curve $\sigma'(k)$ has a minimum

The approximation (15.11) to η_R will obviously be very poor for a value x/t such that $\sigma''(k_r)$ is near to zero for some r unless t is extremely large. It is shown in Jeffreys and Jeffreys, Methods of mathematical physics [3rd ed., Cambridge, 1956, § 17.09] how an Airy function may be used to modify the relevant term in the second summand to give a useful asymptotic expansion for k_r near a zero of σ'' (see also Chester, Friedman and Ursell, Proc. Cambridge Philos. Soc. 53 (1957), 599-611).

If x/t is fixed at a value for which (15.10) has no solution, then for an observer moving with this velocity the disturbance of the surface is very small, for it has been dying out as t^{-1}. The first term of the expansion may, of course, be computed as indicated above. This situation will occur for a disturbance in water of depth h if $x/t > \sqrt{gh}$. It will also occur when surface tension is taken into account for $x/t < \sigma'_{min}$.

The asymptotic expansion (15.11) may also be used in a different fashion. Let us fix our attention upon one value of x and let t increase. Then x/t will decrease and the value $k_1(x/t)$ associated with the point x at a given moment will also change; for pure gravity waves it will increase. The observer stationed at x will then observe waves of continually increasing wave number (decreasing wave length) moving by with phase velocities appropriate to their lengths. The amplitudes at a given instant will depend upon the first factor. The gross outline of the waves will pass the observer at the group velocity appropriate to the wave number present at the moment, and, of course, the amplitude is decreasing as $t^{-1/2}$. In the case of a disturbance on water of depth h, if the observer is initially far from the hump, then even for large enough values of t for the asymptotic expansion to be valid the value of x/t may be greater than $\sqrt{gh}$. Then the observer will see practically no disturbance until the gravest modes begin to reach him. We note again that he must anticipate the arrival of a given wave number by its group velocity, not phase velocity, for it is the former which controls the amplitude. In the case of combined gravity-capillary waves, when t is large enough one will have $x/t < \sigma'_{min}$ and the disturbance will be negligible.

It is also possible to find an asymptotic expansion for $\eta_R(x,t)$ for x/t fixed and large x. It turns out to be the same as (15.11) with $\mathcal{O}(t^{-2/3})$ replaced by $\mathcal{O}(x^{-2/3})$. This expansion allows one, so to speak, to take snapshots of the right-hand end of η_R at different instants of time. If we fix t and let x increase, x/t increases also and $k_1(x/t)$ increases. Thus the wave length decreases as one moves to the right; the observed amplitude will depend upon the first factor. For gravity waves on water of depth h, if x is large enough, $x/t > \sqrt{gh}$ and the disturbance will be small of order x^{-1}.

Finally, we use the asymptotic expansion to investigate the motion of a particular phase of $\eta_R(x,t)$, say a zero, for large t. Such a point will be determined by

$$\alpha(x,t) \equiv k_1 x - \sigma(k_1)t = \text{const.},$$

where, as usual, $k_1 = k_1(x/t)$; solving for x gives $x = x(t)$. One may find $\dot{x}(t)$ from

$$\dot{x}(t) = -\frac{\alpha_t}{\alpha_x} = -\frac{-k_1' \frac{x}{t^2} - \sigma(k_1) + \sigma'(k_1)\frac{x}{t^2}t}{k_1 + k_1'\frac{x}{t} - \sigma'(k_1)k_1'} = \frac{\sigma(k_1)}{k_1}.$$

Thus a particular phase travels with the phase velocity of the harmonic component associated with it at the moment. However, if the group and phase velocities are different, it is then moving at a different velocity from a point just keeping pace with waves of a given wave number. In particular, for gravity waves it is moving faster, hence moves into region of lower wave number and higher velocity and is accelerating. A computation of $\ddot{x}$ bears this out:

$$\ddot{x}(t) = -\frac{1}{t k_1 \sigma''}\left[\frac{\sigma}{k_1} - \sigma'\right]^2,$$

for this is always positive for gravity waves. The right-hand side is, of course, a function of x and t. For deep-water gravity waves the function x(t) may easily be found from the earlier equation:

$$x = \frac{\sigma(k_1)}{k_1}t - \frac{a}{k_1} = 2\sigma'(k_1)t - \frac{a}{k_1} = 2\frac{x}{t}t - \frac{4a}{g}\frac{x^2}{t^2}$$

or

$$x(t) = \frac{gt^2}{4a}.$$

Hence $\ddot{x} = g/2a$ and for large t the acceleration is constant. If the depth is finite, the computation is no longer simple, although it is possible to show that x(t) varies from $x(t) = t\sqrt{gh}$ for a phase associated with $k = 0$ to $x(t) = At^2$ for a phase associated with very large k.

Figure 10 is taken from a paper of Kelvin's [190?], and shows the computed values of $\eta(x,t)$ for an initial displacement given by

$$\eta(x,0) = \frac{\left[1 + (1+x^2)^{1/2}\right]^{1/2}}{2^{1/2}(1+x^2)^{3/2}}\left[2 - (1+x^2)^{1/2}\right]$$

and for $t/\pi^{1/2} = \frac{1}{2}, 1, \frac{3}{2}, 4, 8$ (the units have been chosen so that $g = 4$). The description of the behavior of $\eta_R(x,t)$ outlined in the preceding paragraphs can be easily verified qualitatively by inspection of the successive snapshots of $\eta_R(x,t)$. Green [1909] has shown that if one estimates the wave length at any maximum as double the distance between the two including zeros, then the position is very close to that which would be estimated by using the group velocity [cf. Havelock, 1914, p. 37].

Figure 11 from a report by J. E. Prins [1956; also 1958b] shows measured time histories taken at various distances from the center of an initial rectangular hump of length 2L and height Q in water of depth h for specific values shown in the figure. In general, the features of the motion described above were well verified by this experimental investigation.

We assemble here the expressions for $\sigma(k)$ and $k\sigma'/\sigma$ for a number of cases of water waves.

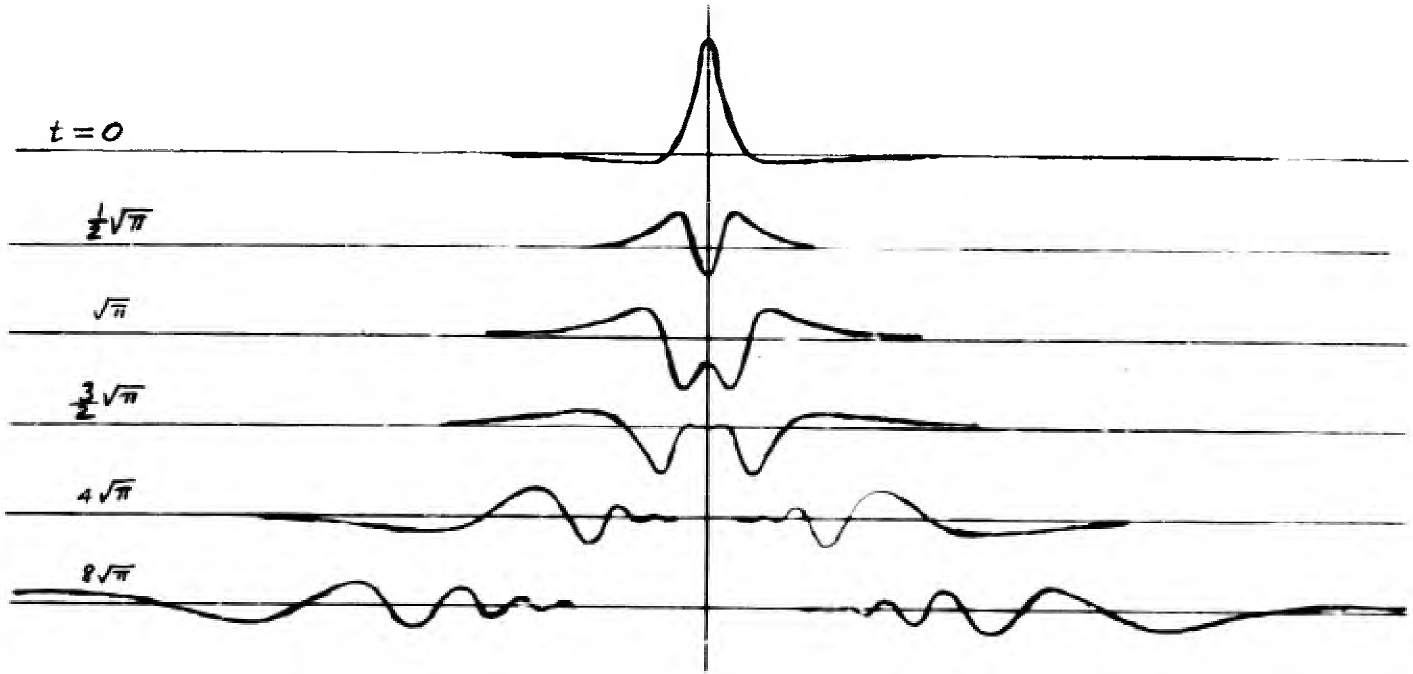


Figure 10

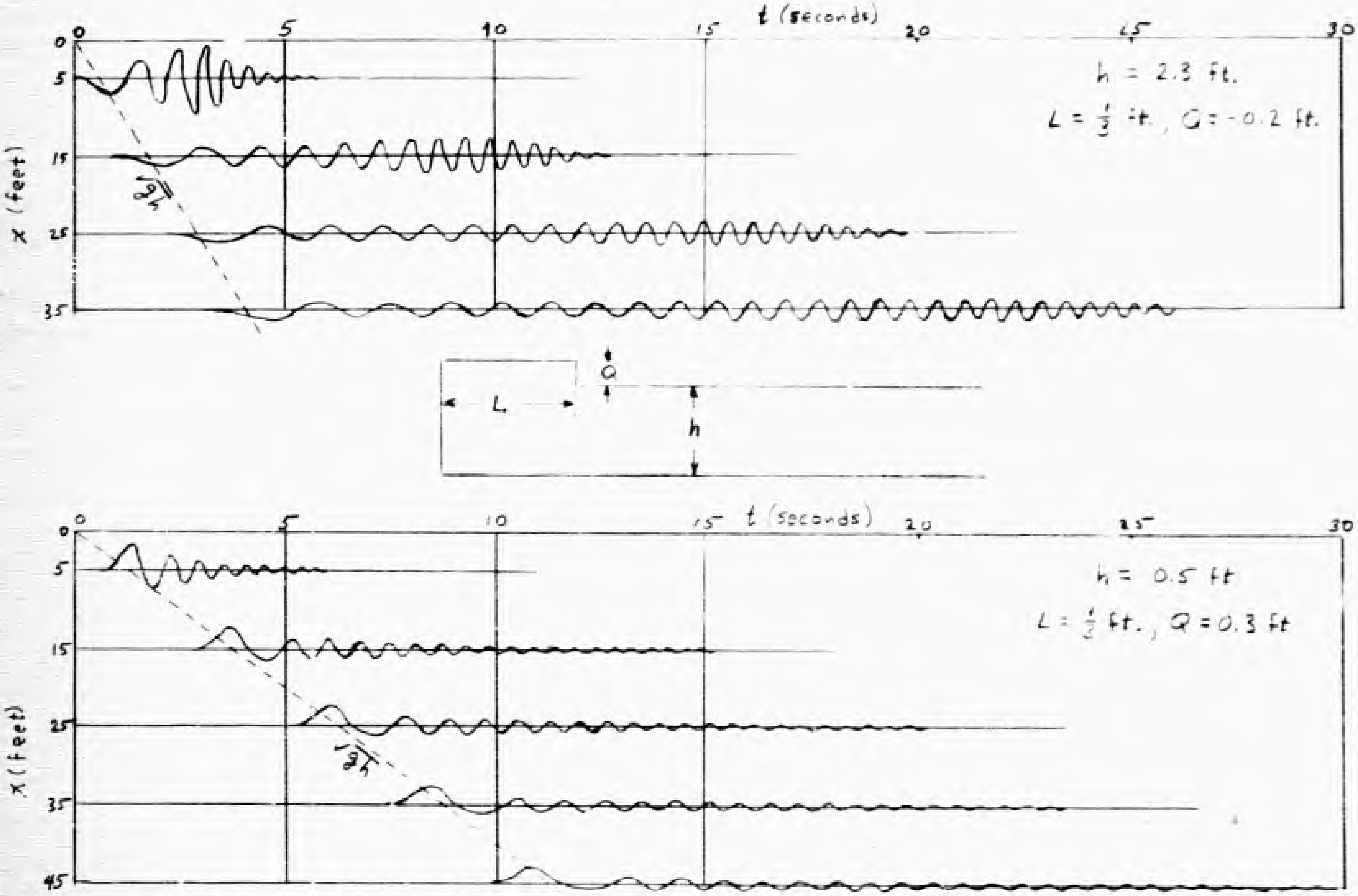

t (seconds)
x (feet)
h = 2.3 ft.
L = 1/3 ft., Q = -0.2 ft.
$\sqrt{gh}$
Q
L
h
h = 0.5 ft
L = 1/3 ft., Q = 0.3 ft

Figure 11

1) Deep-water gravity waves:

$$\sigma(k) = \sqrt{gk}\ , \quad \frac{k\sigma'}{\sigma} = \frac{1}{2}$$

2) Gravity waves at the interface of two fluids, each of infinite vertical extent:

$$\sigma(k) = \sqrt{\frac{\rho_2-\rho_1}{\rho_2+\rho_1}\, gk}\ , \quad \frac{k\sigma'}{\sigma} = \frac{1}{2}\,.$$

3) Gravity waves in water of depth h:

$$\sigma(k) = \sqrt{gk\tanh kh}\ , \quad \frac{k\sigma'}{\sigma} = \frac{1}{2}\left[1 + \frac{2kh}{\sinh 2kh}\right].$$

4) Gravity waves for a layer of thickness d of one fluid over a deep layer of a heavier one:

$$\sigma_1(k) = \sqrt{gk}\ , \quad \frac{k\sigma_1'}{\sigma_1} = \frac{1}{2}\ ,$$

$$\sigma_2(k) = \sqrt{\frac{\rho_2-\rho_1}{\rho_2\coth kd + \rho_1}\, gk}\ ,$$

$$\frac{k\sigma_2'}{\sigma_2} = \frac{1}{2}\left[1 + \frac{2\,kd}{\rho_2\sinh 2kd + \ldots\rho\,\sinh^2 kd}\right].$$

5) Waves at a free surface of a deep fluid with both gravity and surface tension acting:

$$\sigma(k) = \sqrt{gk + \frac{Tk^3}{\rho}}\ , \quad \frac{k\sigma'}{\sigma} = \frac{1}{2}\,\frac{1 + 3Tk^2/\rho g}{1 + Tk^2/\rho g}\,.$$

6) Waves at a free surface of a fluid of depth h with both gravity and surface tension acting:

$$\sigma(k) = \sqrt{\left(gk + \frac{Tk^3}{\rho}\right)\tanh kh}\,,$$

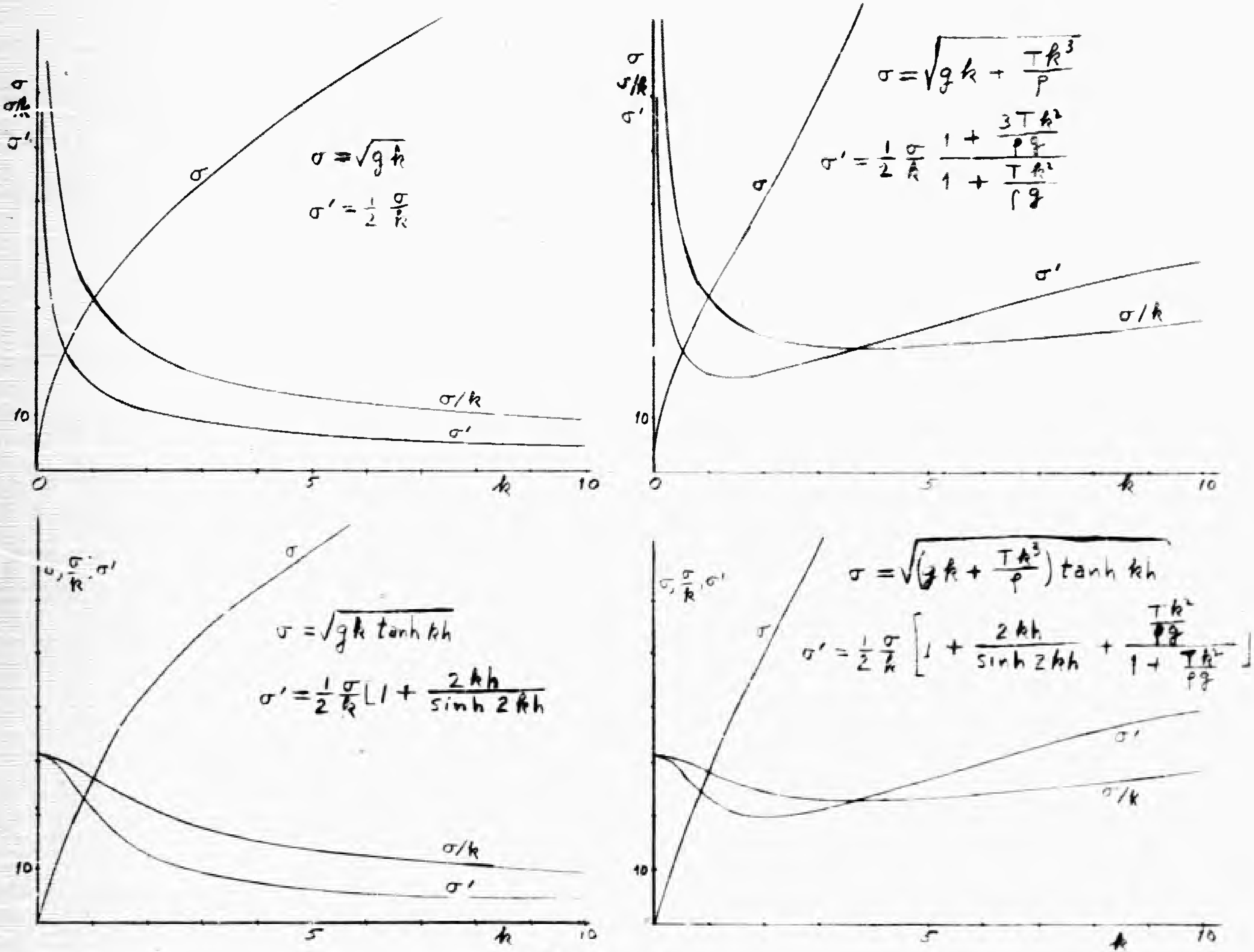
$$\frac{k\sigma'}{\sigma} = \frac{1}{2}\left[1 + \frac{2hk}{\sinh 2hk} + 2\,\frac{Tk^2/\rho g}{1 + Tk^2/\rho g}\right].$$

In cases 1) to 4) σ'' is always negative if $k > 0$. In case 5) it crosses the k-axis at $k = \left[g\rho T^{-1}\frac{1}{3}(2\sqrt{3}-3)\right]^{1/2}$ and becomes positive. In cases 1) to 4) $\sigma' < \sigma/k$ for $k > 0$. In case 5) $\sigma' < \sigma/k$ for $0<k<\sqrt{g\rho/T}$; then σ' crosses σ/k at the minimum of the latter and thereafter remains larger. (Note that σ' always passes through a stationary value of σ/k, passing from beneath to above in going through a minimum, and the reverse at a maximum.) We shall not discuss 6) in detail. For $h > h_c = \sqrt{3T/2\rho g}$, σ/k has a minimum for some k_0, $0 < k_0 < \sqrt{\rho g/T}$, and σ' a minimum to the left of this. For $h \leq h_c$, σ/k is an increasing function, starting at $\sqrt{gh}$ for $k = 0$, and σ' is also increasing, $\sigma' > \sigma/k$ for $k > 0$, $\sigma'(0) = \sqrt{gh}$. Figure 12 shows graphs of σ, σ/k and σ' for 1), 3), 5), and 6) (the scales were chosen for convenience).

One may also take $\lambda = 2\pi/k$ as the independent variable, and then express the phase velocity c and group velocity U as functions of λ. An easy computation shows that

$$\lambda\frac{dc}{d\lambda} = c - U.$$

This equation has a simple interpretation in the geometry of the curve for $c(\lambda)$, as was shown by Lamb [1932, p. 382]: For a given



$\sigma = \sqrt{gk}$
$\sigma' = \frac{1}{2}\frac{\sigma}{k}$
$\sigma = \sqrt{gk + \frac{Tk^3}{\rho}}$
$\sigma' = \frac{1}{2}\frac{\sigma}{k}\frac{1 + \frac{3Tk^2}{\rho g}}{1 + \frac{Tk^2}{\rho g}}$
$\sigma = \sqrt{gk \tanh kh}$
$\sigma' = \frac{1}{2}\frac{\sigma}{k}[1 + \frac{2kh}{\sinh 2kh}]$
$\sigma = \sqrt{(gk + \frac{Tk^3}{\rho}) \tanh kh}$
$\sigma' = \frac{1}{2}\frac{\sigma}{k}[1 + \frac{2kh}{\sinh 2kh} + \frac{\frac{Tk^2}{\rho g}}{1 + \frac{Tk^2}{\rho g}}]$

Figure 12

value of λ , U is the intercept on the vertical axis of the tangent to the $c(\lambda)$ curve at the point $(\lambda, c(\lambda))$. One value of U may correspond to more than one value of λ , as, for example, in the case of gravity-capillary waves. [See Havelock, 1914, § 11.]

15β . The propagation of energy.

It seems intuitively clear that as long as the right-moving part of an initial hump keeps its integrity the energy associated with the motion will in some sense move with the hump. We wish to consider in what sense this is true. We limit ourselves in the following discussion to a single fluid of depth h, where h may become infinite. However, surface tension may act upon the free surface.

We first introduce the notion of energy density for a given value of x. It will be convenient to separate potential, kinetic and surface energy. Let

$$\mathcal{V}(x,t) = \rho g \int_0^{\eta_R(x,t)} y\,dy = \tfrac{1}{2}\rho g\,\eta_R^2(x,t),$$

$$\mathcal{T}(x,t) = \tfrac{1}{2}\rho \int_{-h}^{0} (\Phi_x^2 + \Phi_y^2)\,dy = \tfrac{1}{2}\rho \int_{-h}^{0} (\Phi\Phi_x)_x\,dy + \tfrac{1}{2}\rho\,\Phi\Phi_y(x,0,t), \qquad (15.12)$$

$$\mathcal{S}(x,t) = \tfrac{1}{2} T\,\eta_{Rx}^2(x,t).$$

be the densities of potential, kinetic and surface energies, respectively, where here Φ is the velocity potential corresponding to η_R .

These functions may now be treated in the same way as η_R was in 15α . We may ask for the average position of the distributions of the several densities. They are defined by

$$\bar{x}_V(t) = \int_{-\infty}^{\infty} x\, V(x,t)\, dx \Big/ \int_{-\infty}^{\infty} V(x,t)\, dx,$$

$$\bar{x}_T(t) = \int_{-\infty}^{\infty} x\, T(x,t)\, dx \Big/ \int_{-\infty}^{\infty} T(x,t)\, dx, \tag{15.13}$$

$$\bar{x}_S(t) = \int_{-\infty}^{\infty} x\, S(x,t)\, dx \Big/ \int_{-\infty}^{\infty} S(x,t)\, dx,$$

respectively. Since all three densities are non-negative, one avoids the difficulty met with in defining the average position of η_R. In fact, it is obvious that the definitions of $\bar{x}_R$ and $\bar{x}_V$ coincide, so that the conclusions concerning $\bar{x}_R$ can be applied immediately to $\bar{x}_V(t)$. In particular,

$$\bar{x}_V(t) = \bar{x}_V(0) + \bar{\sigma}'\, t. \tag{15.14}$$

Consider now $\bar{x}_T(t)$. First we note that, from Green's Theorem,

$$\int_{-\infty}^{\infty} T(x,t)\, dx = \tfrac{1}{2}\rho \int_{-\infty}^{\infty} \Phi(x,0,t)\Phi_y(x,0,t)\, dx +$$

$$\lim_{\substack{x_1\to-\infty\\ x_2\to+\infty}} \tfrac{1}{2}\rho \int_{-h}^{0} [-\Phi(x_1,y,t)\Phi_x(x_1,y,t) + \Phi(x_2,y,t)\Phi_x(x_2,y,t)]\, dy.$$

From the assumed square-integrability of η_R, the limit vanishes. Use of the identity $x(\Phi_x^2 + \Phi_y^2) = (x\Phi)_x\Phi_x + (x\Phi)_y\Phi_y - \Phi\Phi_x$ and Green's Theorem gives

$$\int_{-\infty}^{\infty} x\, T(x,t)\, dx = \tfrac{1}{2}\rho \int_{-\infty}^{\infty} x\, \Phi(x,0,t)\Phi_y(x,0,t)\, dx - \lim_{\substack{x_1\to-\infty\\ x_2\to+\infty}} \tfrac{1}{4}\rho \int_{-h}^{0} [\Phi^2(x_2,y,t) - \Phi^2(x_1,y,t)]\, dy$$

$$+ \lim_{\substack{x_1\to-\infty\\ x_2\to\infty}} \tfrac{1}{2}\rho \int_{-h}^{0} [-x_1\Phi(x_1,y,t)\Phi_x(x_1,y,t) + x_2\Phi(x_2,y,t)\Phi_x(x_2,y,t)]\, dy,$$

where again the last two limits vanish. A similar computation shows

$$\int_{-\infty}^{\infty} x^2 T(x,t)\,dx = \frac{1}{2}\rho\int_{-\infty}^{\infty} x^2 \Phi(x,0,t)\Phi_y(x,0,t)\,dx + \frac{1}{2}\rho\int_{-\infty}^{\infty}\int_{-h}^{0} \Phi^2(x,y,t)\,dx\,dy$$

$$+ \lim_{\substack{x_1\to-\infty \\ x_2\to\infty}} \frac{1}{2}\rho\int_{-h}^{0}\left[-x_1^2\Phi(x_1,y,t)\Phi_x(x_1,y,t) + x_2^2\Phi(x_2,y,t)\Phi_x(x_2,y,t)\right]dy$$

$$- \lim_{\substack{x_1\to-\infty \\ x_2\to\infty}} \frac{1}{2}\rho\int_{-h}^{0}\left[-x_1\Phi^2(x_1,y,t) + x_2\Phi^2(x_2,y,t)\right]dy.$$

Collecting these results we have

$$\int_{-\infty}^{\infty} T(x,t)\,dx = \frac{1}{2}\rho\int_{-\infty}^{\infty}\Phi(x,0,t)\,\Phi_y(x,0,t)\,dx,$$

$$\int_{-\infty}^{\infty} x\,T(x,t)\,dx = \frac{1}{2}\rho\int_{-\infty}^{\infty} x\,\Phi(x,0,t)\Phi_y(x,0,t)\,dx,$$

$$\int_{-\infty}^{\infty} x^2 T(x,t)\,dx = \frac{1}{2}\rho\int_{-\infty}^{\infty} x^2\Phi(x,0,t)\Phi_y(x,0,t)\,dx + \frac{1}{2}\rho\int_{-\infty}^{\infty}\int_{-h}^{0}\Phi^2(x,y,t)\,dx\,dy. \tag{15.15}$$

Since from (15.2),

$$\eta_R(x,t) = \frac{1}{2}\int_{-\infty}^{\infty} E(k)\,e^{-i(kx-\sigma t)}\,dk$$

and

$$\Phi(x,y,t) = \frac{1}{2}\int_{-\infty}^{\infty} i\,\frac{\sigma(k)}{k}\,Y(y)E(k)\,e^{-i(kx-\sigma t)}\,dk, \tag{15.16}$$

one finds easily

$$\Phi(x,0,t) = \frac{1}{2}\int_{-\infty}^{\infty} i\,\frac{\sigma(k)}{k}\coth kh\;E(k)\,e^{-i(kx-\sigma t)}\,dk,$$

$$\Phi_y(x,0,t) = \frac{1}{2}\int_{-\infty}^{\infty} i\,\sigma(k)\,E(k)\,e^{-i(kx-\sigma t)}\,dk.$$

One may now apply again, as in section 15α , theorems on Fourier transforms to obtain

$$\int_{-\infty}^{\infty} T(x,t)\,dx = \frac{1}{4}\pi\rho\int_{-\infty}^{\infty} E(k)E^*(k)\,\frac{\sigma^2}{k}\coth kh\,,$$

$$\int_{-\infty}^{\infty} x\,T(x,t)\,dx = \frac{1}{4}\pi\rho\int_{-\infty}^{\infty} i\,E(k)E^{*\prime}(k)\frac{\sigma^2}{k}\coth kh\,dk + \frac{1}{4}\pi\rho t\int_{-\infty}^{\infty} E(k)E^*(k)\,\sigma'(k)\frac{\sigma^2}{k}\coth kh\,dk,$$

$$\int_{-\infty}^{\infty} x^2\,T(x,t)\,dx = \int_{-\infty}^{\infty} x^2\,T(x,0)\,dx + \frac{1}{2}\pi\rho t\int_{-\infty}^{\infty} i\,E(k)E^{*\prime}(k)\,\sigma'\frac{\sigma^2}{k}\coth kh\,dk \qquad (15.17)$$

$$+ \frac{1}{4}\pi\rho t^2\int_{-\infty}^{\infty} E(k)E^*(k)\,\sigma'^2(k)\frac{\sigma^2}{k}\coth kh.$$

If one uses the definition introduced earlier for average of a function of k, one now finds

$$\overline{x}_T(t) = \overline{x}_T(0) + t\,\frac{\overline{\sigma'\sigma^2 k^{-1}\coth kh}}{\overline{\sigma^2 k^{-1}\coth kh}} \qquad (15.18)$$

and a rather unwieldy expression for $\overline{[x - \overline{x}_T(t)]^2}$, similar in character to (15.4). We note that if we are dealing with pure gravity waves, so that $\sigma^2 = gk\tanh kh$, then formulas (15.17) simplify considerably and become identical with those for V. In this case the potential and kinetic energies are equal and propagate with the same velocities.

We may now carry out similar calculations for S(x,t). The corresponding formulas follow

$$\int_{-\infty}^{\infty} S(x,t)\,dx = \frac{1}{4}\pi T\int_{-\infty}^{\infty} k^2 E(k)E^*(k)\,dk,$$

$$\int_{-\infty}^{\infty} x\,S(x,t)\,dx = \frac{1}{4}\pi T\int_{-\infty}^{\infty} i\,k^2 E^{*\prime}(k)E(k)\,dk + \frac{1}{4}\pi T t\int_{-\infty}^{\infty} k^2\sigma'(k)\,E(k)\,E^*(k)\,dk,$$

$$\int_{-\infty}^{\infty} x^2 S(x,t)\,dx = \int_{-\infty}^{\infty} x^2 S(x,0)\,dx + \frac{1}{2}\pi T t\int_{-\infty}^{\infty} k^2\sigma'\,E^{*\prime}(k)E(k)\,dk \qquad (15.19)$$

$$+ \frac{1}{4}\pi T t^2\int_{-\infty}^{\infty} k^2\sigma'^2\,E(k)\,E^*(k)\,dk,$$

and

$$\bar{x}_s(t) = \bar{x}_s(0) + t\,\frac{\overline{k^2\sigma'}}{\overline{k^2}} \tag{15.20}$$

and again a formula for $\overline{[x - \bar{x}_s(t)]^2}$ similar in character to (15.4).

One should note that the total potential, kinetic and surface energies associated with $\eta_R(x,t)$ each remain constant in time. If $T \neq 0$, then the mean positions of the three energy densities propagate with different velocities, each velocity being an average, in some sense, of σ'. If one considers a wave packet (15.5), then as the width 2ε of the band of wave numbers approaches zero the velocity of propagation of the individual energy densities will each approach $\sigma'(k_0)$, the group velocity.

Consider now the total energy density,

$$\mathcal{E}(x,t) = \mathcal{V}(x,t) + \mathcal{T}(x,t) + \mathcal{S}(x,t).$$

Making use of the form of $\sigma(k)$,

$$\sigma^2(k) = (gk + Tk^3/\rho)\tanh kh,$$

one finds

$$\int_{-\infty}^{\infty} \mathcal{E}(x,t)\,dx = \tfrac{1}{4}\pi\rho\int_{-\infty}^{\infty}\left[g + \frac{\sigma^2}{k}\coth kh + \frac{T}{\rho}k^2\right]E(k)E^*(k)\,dk$$

$$= \tfrac{1}{2}\pi\int_{-\infty}^{\infty}[g\rho + Tk^2]\,E(k)E^*(k)\,dk,$$

$$\int_{-\infty}^{\infty} x\,\mathcal{E}(x,t)\,dx = \tfrac{1}{2}\pi\int_{-\infty}^{\infty}[g\rho + Tk^2]\,iE^{*\prime}(k)E^*(k)\,dk + \tfrac{1}{2}\pi t\int_{-\infty}^{\infty}\sigma'(k)[g\rho + Tk^2]EE^*\,dk,$$

$$\int_{-\infty}^{\infty} x^2\mathcal{E}(x,t)\,dx = \int_{-\infty}^{\infty} x^2\mathcal{E}(x,0)\,dx + \pi t\int_{-\infty}^{\infty}\sigma'[g\rho + Tk^2]\,iE^{*\prime}E\,dk \tag{15.21}$$

$$+ \tfrac{1}{2}\pi t^2\int_{-\infty}^{\infty}\sigma'^2[g\rho + Tk^2]\,EE^*\,dk,$$

and

$$\bar{x}_E(t) = \bar{x}_E(0) + t\,\frac{\overline{\sigma'[\rho g + Tk^2]}}{\overline{\rho g + Tk^2}}\,. \tag{15.22}$$

At any instant t half of the total energy is kinetic energy and the other half is divided between potential and surface energy.

There is another way of considering the energy transported by surface waves which, at first glance, is different from the preceding treatment. Consider a fixed plane x = constant. Then from the results in § 8 one may compute the rate at which energy is being transported through this plane, the so-called <u>energy flux</u>. Let us denote it by $\mathfrak{F}(x,t)$. After appropriate linearization, formula (8.10) gives

$$\mathfrak{F}(x,t) = -\int_{-h}^{0} \rho\, \Phi_t(x,y,t)\Phi_x(x,y,t)\,dy - T\eta_t(x,t)\eta_x(x,t). \tag{15.23}$$

The expression for the flux has an advantage over the expressions for mean positions considered above in that no strong restrictions upon η are required for it to exist. In fact, it can be computed for a single harmonic wave

$$\eta = A \sin(kx - \sigma t). \tag{15.24}$$

With

$$\Phi = -A\,\frac{\sigma}{k}\,\frac{\cosh k(y+h)}{\sinh kh}\cos(kx - \sigma t),$$

one finds by a straight forward calculation

$$\mathfrak{F}(x,t) = A^2 T k\sigma \cos^2(kx-\sigma t) + A^2\rho\,\frac{\sigma^3}{2k^2}\coth kh\left[1 + \frac{2kh}{\sinh 2kh}\right]\sin^2(kx-\sigma t).$$

Averaging over a wave length (or over a period, it makes no difference which), one finds

$$\mathcal{F}_{av} = A^2 \frac{1}{4} \frac{\sigma}{k} \left\{ 2Tk^2 + \sigma^2 \rho \frac{\coth kh}{k} \left[1 + \frac{2kh}{\sinh 2kh} \right] \right\}$$

$$= \frac{1}{2} A^2 (g\rho + Tk^2) \sigma'(k). \quad (15.25)$$

Thus the group velocity enters again in connection with energy propagation, even though no "group" is present. The energy density and average energy per wave length for (15.24) are

$$\mathcal{E}(x,t) = A^2 \left\{ \frac{1}{2} \rho g \sin^2(kx-\sigma t) + \frac{1}{2} T k^2 \cos^2(kx-\sigma t) + \frac{1}{4} \rho \frac{\sigma^2}{k} \coth kh \left[1 - \frac{2kh}{\sinh 2kh} \cos 2(kx-\sigma t) \right] \right\},$$

$$\mathcal{E}_{av} = \frac{1}{2} A^2 (g\rho + Tk^2). \quad (15.26)$$

If one is dealing with a composite wave, averaging over a wave length is possible only if the resulting wave is periodic. However, even without this restriction, one may compute both the average flux and average energy per unit length from

$$\mathcal{F}_{av} = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} \mathcal{F}(x,t)\,dx,$$

$$\mathcal{E}_{av} = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} \mathcal{E}(x,t)\,dx. \quad (15.27)$$

Then if a composite wave propagating to the right is given by

$$\eta(x,t) = \sum_{j=-\infty}^{\infty} E_j e^{-i(k_j x - \sigma_j t)}$$

$$= \sum_{j=1}^{\infty} a_j \cos(k_j x - \sigma_j t) + b_j \sin(k_j x - \sigma_j t), \quad (15.28)$$

with

$$\Phi = \sum_{j=-\infty}^{\infty} i E_j \frac{\cosh k_j(y+h)}{\sinh k_j h} \frac{\sigma_j}{k_j} e^{-i(k_j x - \sigma_j t)},$$

where $E_j = E^*_{-j} = \frac{1}{2}(a_j + b_j)$, $k_{-j} = -k_j$, $\sigma_j = \sigma(k_j) = -\sigma_{-j}$, one finds

$$\begin{aligned} \mathcal{E}_{av} &= \frac{1}{2}\sum_{-\infty}^{\infty} |E_j|^2 \left[T k_j^2 + \rho g + \rho \frac{\sigma_j^2}{k_j} \coth k_j h\right] \\ &= \sum_{-\infty}^{\infty} |E_j|^2 [\rho g + T k_j^2] = \frac{1}{2}\sum_{1}^{\infty} (a_j^2 + b_j^2)[\rho g + T k_j^2] \end{aligned} \tag{15.29}$$

and

$$\begin{aligned} \mathcal{F}_{av} &= \sum_{-\infty}^{\infty} |E_j|^2 \left\{T k_j \sigma_j + \frac{\sigma_j^3}{2 k_j^2} \coth k_j h \left[1 + \frac{2 k_j h}{\sinh 2 k_j h}\right]\right. \\ &= \sum_{-\infty}^{\infty} |E_j|^2 [\rho g + T k_j^2] \sigma_j' . \end{aligned} \tag{15.30}$$

In order to obtain these relatively simple formulas in which the contributions from the individual harmonics are isolated, it is essential that the averages be taken. Otherwise, for $\mathcal{E}(x,t)$ or $\mathcal{F}(x,t)$ one obtains a complicated double summation, and the role of the group velocity is not apparent.

A similar analysis may be carried through for the right-moving initial hump (15.16). However, an average of either $\mathcal{F}$ or $\mathcal{E}$ computed according to (15.27) would vanish. Instead we take the total flux and total energy, respectively:

$$\mathcal{F}_{total} = \int_{-\infty}^{\infty} \mathcal{F}(x,t)\, dx \quad , \quad \mathcal{E}_{total} = \int_{-\infty}^{\infty} \mathcal{E}(x,t)\, dx \tag{15.31}$$

The resulting formulas are analogous to (15.29) and (15.30):

$$\begin{aligned} \mathcal{E}_{total} &= \frac{1}{2}\pi \int_{-\infty}^{\infty} [g\rho + T k^2]\, E(k) E^*(k)\, dk , \\ \mathcal{F}_{total} &= \frac{1}{2}\pi \int_{-\infty}^{\infty} \sigma'(k) [g\rho + T k^2] E(k) E^*(k)\, dk . \end{aligned} \tag{15.32}$$

If the last result is applied to a narrow wave band, such as (15.5), then one finds the limiting relationship

$$\lim_{\varepsilon \to 0} \frac{\mathcal{F}_{total}}{\mathcal{E}_{total}} = \sigma'(k_0).$$

In the first method of treating the propagation of energy, i.e. in terms of the motion of the mean position of the energy density, it was not surprizing that σ' should appear, for it is a familiar property of Fourier transforms that taking the derivative of the transform is associated with multiplying the function by the variable. Thus, if

$$g(k) = \int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx,$$

then

$$g'(k) = \int_{-\infty}^{\infty} i x\, f(x)\, e^{ikx} dx.$$

In the cases considered above the transform contained $e^{i\sigma t}$ as a factor, and the derivative contained $\sigma' t$ in one summand. However, the appearance of σ' in the formulas for $\mathcal{F}_{av}$ or $\mathcal{F}_{total}$ seems in some ways coincidental: One makes a calculation, and after gathering and manipulating terms discovers that a certain combination of them indeed contains σ'. That this is not really coincidence is indicated by the following theorem for the case (15.16):

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} x\, \mathcal{E}(x,t)\, dx = \mathcal{F}_{total}. \qquad (15.33)$$

It may be proved as follows. From the definition of $\mathcal{E}(x,t)$

$$\int_{-\infty}^{\infty} x\,\mathcal{E}(x,t)\,dx = \int_{-\infty}^{\infty} x\left[\tfrac{1}{2}\rho g\eta^2 + \tfrac{1}{2}T\eta_x^2 + \tfrac{1}{2}\rho\int_{-h}^{0}(\Phi_x^2+\Phi_y^2)\,dy\right]dx.$$

Hence

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} x\,\mathcal{E}(x,t)\,dx = \int_{-\infty}^{\infty} x\left[\rho g\eta\eta_t + T\eta_x\eta_{xt} + \rho\int_{-h}^{0}(\Phi_x\Phi_{xt}+\Phi_y\Phi_{yt})\,dy\right]dx.$$

Integrating the second and third terms by parts and taking account of the assumed behavior of η and Φ at $\pm\infty$, one finds

$$\int_{-\infty}^{\infty} x\left[\rho g\eta\eta_t - T\eta_{xx}\eta_t + \rho\int_{-h}^{0}(\Phi_y\Phi_{yt}-\Phi_{xx}\Phi_t)\,dy\right]dx$$

$$- \int_{-\infty}^{\infty}\left[T\eta_x\eta_t + \rho\int_{-h}^{0}\Phi_x\Phi_t\,dy\right]dx.$$

Since $\Phi_{xx} + \Phi_{yy} = 0$, one may express the third summand in the first integral as

$$\rho\int_{-h}^{0}(\Phi_y\Phi_{yt}+\Phi_{yy}\Phi_t)\,dy = \rho\int_{-h}^{0}(\Phi_y\Phi_t)_y\,dy = \rho\Phi_y(x,0,t)\Phi_t(x,0,t) = \rho\eta_t\Phi_t.$$

Hence the first integral may be written

$$\int_{-\infty}^{\infty} x\eta_t\left[\rho g\eta - T\eta_{xx} + \rho\Phi_t(x,0,t)\right]dx,$$

which vanishes, since the term in brackets is just the dynamical boundary condition at the free surface. The second integral above is just $\mathcal{F}_{total}$, so that (15.33) is proved.

A similar line of reasoning allows one to establish the following relation between $\mathcal{E}$ and $\mathcal{F}$:

$$\frac{\partial\mathcal{E}(x,t)}{\partial t} = -\frac{\partial\mathcal{F}(x,t)}{\partial x}, \qquad (15.34)$$

essentially an expression of the conservation of energy. Equation (15.33) may also be derived from (15.34) by writing the latter in the form

$$\frac{\partial(x\mathcal{E})}{\partial t} = -\frac{\partial(x\mathcal{F})}{\partial x} + \mathcal{F}$$

and integrating.

Although (15.33) may explain the presence of σ' in the energy flux for a continuous spectrum and finite total energy, one is still left with the apparently paradoxical situation that even for (15.24), when only one frequency is present, σ' enters into the expression for $\mathcal{F}_{av}$. One would expect the occurrence of σ' only if one were dealing not only with a specific value k but also with neighboring values. There is no useful analogue to (15.33) for the discrete spectrum, because there is no Fourier integral to connect in a natural way the mean position of a hump with σ'. However, if one approximates (15.24) or (15.28) by considering only the segment of η between -L and L and taking $\eta = 0$ outside this segment, then one has approximated η by η_L, where the latter has a continuous spectrum and finite energy. For η_L it is reasonable that $\sigma'(k)$ should enter into the energy propagation. The definitions adopted for $\mathcal{F}_{av}$ and $\mathcal{E}_{av}$ in (15.27) reflect this approximation of η by η_L and then a passage to the limit in such a way as to keep these quantities finite. Thus it is perhaps not surprizing after all that σ' has entered into the computation of $\mathcal{F}_{av}$, for the method of averaging $\mathcal{F}$ and $\mathcal{E}$ is such that one replaces the discrete spectrum by a continuous one and then takes a limit. A different explanation of this paradoxical situation has been given by Rayleigh

[Theory of sound, vol. I, p. 479]; generally, it seems to be overlooked.

One should note that the definitions of velocity of propagation of mean positions of humps and energy distribution for finite total energy and of total or average energy flux all retain meaning even if the boundary condition at the free surface has not been linearized. The comparative simplicity of the formulas when the boundary condition is linearized and the occurrence in them of σ' both result from the special form of the spectrum, namely $E(k,t) = E(k,0)e^{i\sigma(k)t}$, and the applicability of properties of Fourier transforms of convolutions.

For further information one may consult the monograph of Havelock [1914] already cited, papers by Bourgin [1936], Rossby [1945, 1947], Eckart [194_], Broer [1951], and Poincelot [19_3, 1954], Jeffreys and Jeffreys, Methods of mathematical physics [3rd ed., Cambridge, 1956, pp. 511-518] and standard texts such as Lamb [1932, §§ 236, 237, 240, 241] and Kochin, Kibel' and Roze [1948, ch. 8, § 8].